Children's Development of Place Value & Base Ten Understanding: Building a Multiplicative Rate of Ten

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What is Place Value?

It has long been thought that if a child can name the value of a digit in its particular location that the child understood place value. If a child aligned the columns of digits correctly in a vertical multidigit computation problem in addition, subtraction, or multiplication that child understood place value. But place value is far more sophisticated of a mathematical concept than these features. A child's understanding of place value and the base ten system[1] is built upon a broad conceptual foundation of number size, number relations, flexibly decomposing and reconfiguring numbers, and thinking multiplicatively. The later is key as place value is inherently a multiplicative relation.

What makes multiplication different than addition, and thus how does this relate to place value? Multiplication requires the making of units of units, then using the new composite unit for more efficient measurement or quantification. As a result, multiplication requires an ability to think simultaneously across units (Kamii, 2000). For every new composite unit that is made, one has to coordinate the accumulating sub-units of the composite unit. (See Figure 1.)

Multiplication is unit transforming. If I have 3 bags of bagels with 10 bagels per bag, I have a total of 30 bagels. The bags dissolve in the transformation. (See Figure 2.) In contexts of rates, units such as miles per hour transform into new units such as speed. A robust understanding of place value requires a capacity to think multiplicatively as it is built on a foundation of rate, ratio, and proportion. This is far more sophisticated than the single skill of naming the value of a digit based upon its location.

Building Place Value Concepts – From Additive to Multiplicative Thinking

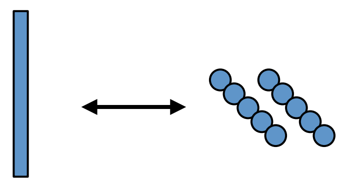
Observing Solution Strategies

Consider the following students' solution strategies to this problem given in a first grade classroom. *You have a pile of 42 pop tabs on the table. How many piles of ten can you make?*

Daniel's strategy: Daniel makes 42 tallies in his math journal. He circles ten tally marks to make one group and continues to do so until no more groups of ten can be made. He counts the groups and says, "4 piles of 10 pop tabs."

Rose's strategy: Rose uses her math journal as well. She draws a circle and puts 10 dots in it. She draws another circle and puts 10

Figure 1 – Place Value as a Multiplicative Proportion



For every one ten there are ten ones
1:10
For every ten ones there is one ten
10:1

Figure 1a
Making a Unit of units

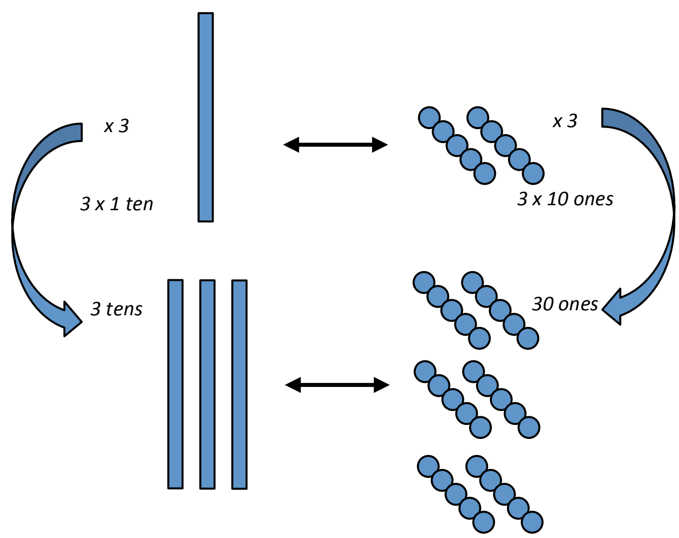


Figure 1b
Coordination of Units – Coordination of a Ratio – Unit Transformation

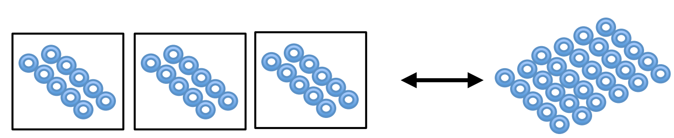


Figure 2
Unit transforming
3 bags with 10 bagels per bag ←→ 30 individual bagels
The unit of bags dissolves when finding the total number of individual bagels

[1] While some researchers distinguish between the terms place value and base ten understanding, in this article they are being used synonymously.

dots in that one as well. She continues doing this until she has 4 circles and two loose dots not in a circle. She verbalizes her counting, "10, 20, 30, 40, 41, 42," and says, "4 groups of 10."

Diego's strategy: Diego wrote the number 10 in his journal and says, "10." He writes another 10 and says, "20," another 10, "30," another 10, "40." He then makes two tally marks for 41 and 42. His answer is, "4 piles of 10."

Rory's strategy: Rory talks out loud saying, "Well, 20 pop tabs would be 2 piles of ten, so 40 pop tabs would be another two piles, and there would be two left over, so 4 piles of ten."

Cindy's strategy: Cindy immediately says, "4, because 4 tens are 40."

Daniel in his mind sees 42 as a pile of ones. Rose is beginning to understand that collections of 10 can be assembled and counted. Diego sees ten as a countable unit. Rory is deriving his answer by thinking relationally about what he knows about 20 being 10 + 10 and 40 being 20 + 20. He is working mentally. Cindy has developed a multiplicative understanding that allows her to see the tens and ones relations mentally, developing the multiplicative understanding that 42 = 4 x 10 + 2.

These progressions reflect the children's use of direct modeling (Daniel and Rose), counting (Diego), deriving (Rory), or number level and understanding the multiplicative place value relations (Cindy). This progression is not linear. Increase the numbers to a three-digit combination and Rory and Cindy may have to use a counting or even a direct modeling strategy to negotiate the more complex relations. What the progression does show is how children at young ages can begin to think about number in various levels of sophistication and how over time come to understand that concept of place value as a multiplicative relations.

Magnitude of Number

Another aspect of place value is related to the magnitude of one number compared to another. The act of counting large quantities helps young children to understand that a pile of 36 is larger than a pile of 24. Letting young children count and count frequently helps this core number sense grow. To consider a new strategy, the current strategy of counting by ones has to become regarded as no longer efficient. If numbers are kept small, there is no reason in the minds of children to consider a new counting strategy. In the classroom having simultaneous small groups of children counting large quantities of numbers can lead to a noisy cacophony where students lose their count and have to go back to the beginning to start all over. Guided conversations around this frustration of needing to start over because one has lost ones place can lead to discussions around ways to organize the objects so one more easily remembers what is in each smaller pile. While students may suggest placing such objects into piles of fives, tens, or twenties, it does not take long for most children to gravitate to placing the objects into piles of ten. Summary discussions with students over these efficiencies in the number system assists in building visual models about how and why the number system works the way it does.

Number Relationships – Using the Addition & Subtraction Problem Types to Develop Place Value

The problem types, with strategically selected number combinations, can foster conversations within a classroom about how numbers are configured and decomposed in terms of place value. Think about the following problem and subsequent number selection.

You have 10 rocks in your pail. How many more rocks do you need in your pail to have 14?

The instructional conversation is around how 14 is the same as 10 + 4. Consider these other problem types with number combinations that explore how a number is built around place value ideas. Each problem allows the child to consider the relationships within a number from a slightly different perspective.

You have 10 rocks in your pail. I have 14? How many more do I have than you? (Compare, difference unknown)

You had 14 rocks. You gave 4 to your friend. How many do you have left? (Separate, result unknown)

You had some rocks in your pail. You found 4 more rocks. Now there are 14 rocks in your pail. How many did you start with? (Join, start unknown)

In each of these addition or subtraction problems, *the choice of numbers* allows a child to explore how a number is built, how they compare, and how they are pulled apart in terms of place value. As the teacher, the problem types you select, and the numbers you choose are ways you shape mathematical conversations in the classroom.

Re-Unitizing a Number

Now consider the number 783. Now answer for yourself the following questions.

How many tens are in the number 783?

How many tens are in the tens place in 783?

Are the answers to these questions the same? No, they are not! And they are not trick questions, either. The former is referencing the whole number but is often misinterpreted to be asking the more specific location information asked by the latter. The number 783 contains 783 ones, 78 whole groups of ten, and 7 whole groups of one hundred. A robust understanding of place value requires not only knowing the answer to the second question, the one most typically asked, but just as importantly requires the fluid capacity to *re-unitize* a number across places. Mathematically this is what we do when decomposing or reconfiguring numbers particularly in subtraction and division. It is also the basis of working with decimals. When I want find the answer to 4 ÷ 8, seeing 4 ones as equivalent to 40 tenths makes the division easier to execute. When you read in an article the number '2.3 million,' the number is a dehydrated version, so to speak, of the original quantity of ones, 2,300,000. If I take the same number and reference the unit to hundred thousands, the answer would be 23 hundred thousands.

These are all examples of where the need to look at a number *across places* is just as essential as looking at a number *within a place.* Children need to develop the capacity to answer both questions.

Place Value as a Rate of Ten

When a child says "Forty is four tens," that translates mathematically to be 40 = 4 x 10. If I need to convert 2.3 million to an equivalent unit of hundred thousands, the conversion is a magnitude of ten resulting in 23 hundred thousands. This is what is called the rate of ten and is foundational to understanding the base ten system.

The rate of ten can also be explored with mathematical tasks such as 30 x 300. Most of us growing up, and many school curricula today, encourage the multiplication of 3 x 3 then "count the zeros." This is a surface level treatment of an important mathematical idea. Ask yourself this question, what do you do *mathematically* to the 30 and the 300 to isolate the 3 in each number? Drawing attention to the idea that one is factoring 30 into 3 x 10 and 300 into 3 x 100 exposes the multiplicative rate of ten. Consider the following:

$$\begin{aligned} 30 \times 300 &= (3 \times 10) \times (3 \times 100) \\ &= 3 \times (10 \times 3) \times 100 \\ &= 3 \times (3 \times 10) \times 100 \\ &= (3 \times 3) \times (10 \times 100) \\ &= 9 \times 1000 \\ &= 9000 \end{aligned}$$

The factors of ten reveal how the rate of ten is underpinning the base ten system. Just "counting zeros" aids nothing in the development of this most important idea. Here the decomposition process of factoring and the commutative and associative properties are accessed to determine the calculation with a level of mathematical understanding that nurtures a robust understanding of the place value/base ten system.

The Role of Language

Studies have shown that children from Chinese based languages develop place value concepts sooner and more securely than Western based languages (Fuson, 1990). Language it is felt plays a significant role in these differences. While in English we have twelve unique names for the first 12 numbers in our system, Chinese languages literally start describing 11 as 'ten-one', 12 as 'ten-two', 20 as 'two-ten', and 21 as 'two-ten-one.' In a longitudinal study here in the United States, children who used what is referred to as 'invented strategies,' strategies such as tens & ones, incremental, or compensation, developed place value concepts sooner and more securely than those who relied upon the standard algorithms (Carpenter, et al., 1998). Students who use these 'invented' strategies consistently talk in value rather than in digits. When adding 56 and 37, they decompose the numbers and say 50 + 30 equals 80 rather than 5 + 3 equals 8. Developmental psychology has long noted that young children can only think about one attribute at a time. Thus if a child uses single digit language, he or she tends to believe that is the value of the part of the number they are working with even if the digit is in the tens place (Kamii, 2000). Supporting children to speak in value undergirds their place value development even if when they use the traditional algorithms.

Using Multiplication, Measurement Division & Multi-step Problems

The number 42 can be decomposed into many different combinations. If the focus is on its place value components, however, this can be represented in its additive form (40 + 2) and its multiplicative form (4 x 10 + 2). To help young children build an understanding of place value, and develop a rate of ten, multiplication and measurement division become key operations by which these relationships can be explored.

Typically, multiplication is formally introduced as an operation in the third grade. In some second grade curriculum, multiplication is used to help students skip count by twos, fives and tens with such activities as, "How many eyes are there in the classroom?" or "How many fingers do we have in the classroom?" The emphasis of these activities is the rote skip counting sequence rather than having students explore multiplication as an operation.

Research shows that over 87% of kindergartners in the spring of the year were able to use a correct strategy to solve a multiplication problem, and 74% used a correct strategy to solve a measurement division problem (Carpenter, et al., 1993). The reason is that the structures are so easily modeled. Given contexts that students relate to and the ability to freely model the contexts, multiplication and division is within the command of primary students. While these primary students are thinking additively rather than in multiples, the structure of multiplication and measurement division, especially when ten is the organizing quantity, allow students to explicitly explore place value relationships within a multidigit number.

Multiplication

Using a multistep equal grouping context of multiplication, and addition where the problem context emphasizes 10 as the number within the countable element, the teacher can assist students in building a tens and ones understanding. Consider the following task:

> *The coach has 3 bags of soccer balls. Each bag has 10 soccer balls inside. The coach also has 4 loose balls. How many soccer balls does the coach have for the team to use?*

Celeste takes unit blocks and builds three piles of blocks with ten blocks in each pile. She puts 4 blocks in a fourth pile. She then counts all the blocks starting from one. 34 is her answer.

Eshan takes three ten rods and 4 unit blocks and lays them on the table. He counts each rod off by ones as well as the unit blocks. His answer is 34 soccer balls.

Travis does the same thing but counts the blocks saying, "10, 20, 30, 31, 32, 33, 34. The answer is 34."

Selena reads the problem and says to herself, "10, 20, 30, 31, 32, 33, 34. The answer is 34."

Christie says immediately "34 because 3 tens are 30 plus 4 is 34."

For Celeste, 34 is a collection of ones to be organized into tens and counted by ones. Eshan is able to build in tens and ones with tools but still must count by ones. This is a transitional point you will see students go through; they can think in tens, or they can think in ones, but they can't think simultaneously in tens and ones.

Travis, while still a direct modeler can do what Eshan has not been able to do. He can build in tens and use the tens as a countable object. Selena has moved beyond Travis. She is able to abstract the sets in her head and, using fingers to keep track, count on the tens. Christie has developed the tens and ones place value relationship at an automatic level.

In each case the students were successful solving the problem. Yet, in each case a different level of base ten understanding was in evidence. It is working through this type of problem context and the conversations that are drawn out by the teacher during formal and informal sharing that students come to understand the base ten system. Multiplication, with an emphasis on 10, is an important instructional tool for a teacher to draw upon.

Measurement Division

To children, division creates two distinct structures; one where you know the number of groups in the whole (partitive division) but not the equal amount within each group, and the other where you know the quantity of items within each of the subdivided sets (measurement division) but not the number of groups to be made with those equal quantities. Try the following two questions with your students and watch the difference in how the solution strategies differ.

I have 50 cookies. I want to put them onto 10 plates so that each plate will have the same number of cookies on it. How many will be on each plate?

I have 50 cookies. I want to put them on plates so that there are 10 cookies on each plate. How many plates will I need for my cookies?

The first problem is partitive division. A child will get out the 50 objects and parcel them out typically one by one into 10 groups. There may be some skip counting or it may come through a trial and error process. The second problem, however, has the child getting out 50 blocks and, in clusters of ten, parcel them out until all blocks are distributed. Any skip counting emphasis is specifically centered on ten because ten is the known quantity. This is why measurement division can be an important tool to assist children in exploring 10 as a countable unit even if they are still direct modelers.

The very first problem demonstrated in this article, *You have 42 pop tabs... How many piles of 10 can you make?*, provides a case where children will organize and count by tens. The direct modeler who organizes only in ones will collect 42 blocks, group into clusters of 10 and then count the number of groups. The direct modeler who is beginning to organize in tens will use four ten rods and 2 unit blocks to make 64 then count the rods. The counter will go, "10, 20, 30, 40, 4 boxes," typically using fingers to keep track of the count. Finally, like Cindy at the beginning of this article, the answer is "4 boxes because 40 is four tens."

The very act of organizing into groups of 10 will lead to classroom conversation that in 42, 4 boxes of 10 can be made. That takes care of 40 of the pop tabs. The ten-ness is explicitly explored by the very nature of the context and the numbers the teacher chose to place into the problem.

Partitioning into tens and ones

As students begin to master the tens and ones relationship, one of the next efficiencies teachers wish students to explore is how to partition a number and use the place value to efficiently solve the problem.

I have 9 dozen eggs. How many individual eggs is that?

Marianne quickly says, "9x10=90; 9x2=18 so the answer is 108." Using the teen numbers (11-19) becomes a number set that students can easily partition and use their budding place value understanding about tens and ones to solve the problem quickly and efficiently. If students have explored ten actively as a class, one fact students quickly come to learn early on is that ten tens make one hundred: 10x10=100. Building upon this known fact allows students to solve a problem like the following:

I have 10 dozen eggs. How many individual eggs is that?

William works on paper writing, "10x10=100; 10x2=20 so the answer is 120." William, a second grader, is coming to understand that numbers can be broken into combinations of addends, the parts multiplied, then added to find the total. This is the distributive property of multiplication over addition. His time as a first grader modeling and skip counting large quantities of objects into groups of ten, his solving problems that had him reflect on how numbers like 12 can be decomposed into its place value components of 10 + 2, and his solving of multiplication and measurement division problems where ten was the organizing unit allowed him eventually to solve a task such as 10 x 12 with ease.

Multiplication as '*n-times as many*'

While equal grouping contexts are easy entry for students to engage with multiplication, it is not the only construct for conceptualizing multiplication. Multiplication as *n-times as many* is another construct of the operation that leads to reasoning explicitly about the scaled proportions involved with the coordination of units. The *n-times as many* reasoning is connected to the multiplicative reasoning needed when working with ratios, proportions, and fractional relations. It is this conceptualization of multiplication that students are moving towards as they approach middle school mathematics. Reasoning about the rate of ten, in the quest to develop a robust understanding of place value, is an early entry point in the elementary student's learning where developing this sense of scale can occur. Therefore, place value is a logical place to initiate the capacity to think in this manner.

Reasoning With Ten & Multiplies of Ten

As William incorporates the knowledge that ten tens (10 x 10) are one hundred, reasoning with multiples of ten becomes an area for explicit exploration. An aspect of thinking multiplicatively is to think in scale. Consider the following episode from a third grade classroom:

> The teacher writes the following on the board and says *Do you believe me that there are ten tens in 100? [Gives time for student acknowledgement. Note no operation symbol is present. This is intentional]*:
>
> 10 10 100
>
> If that is true, how many tens are in 300? [Writing underneath the other…]
>
> 10 10 100
> ____ 10 300
>
> Listen to the levels of reasoning that are possible.
>
> Gisela: Ten tens are in one hundred. Twenty tens are in two hundred. So, thirty tens are in three hundred. The answer is 30 tens.
>
> Daniel: Well, three times one hundred is three hundred, so there should be three times as many tens. The answer is 3 x 10 which is 30 tens.

Beginning with combinations of "times three" or "times four" the students are able, if necessary, to use additive scaling to determine the answer. With the variety of conversations and multiple exposures to such tasks, students over time build the capacity to think multiplicatively and recognize the scale factor involved. Starting each time with a relationship that the students can trust or quickly confirm, such as 10 x 10 or 10 x 100, the maximum numbers of students are able to access the conversation and begin to engage in the scaling process.

Reasoning around factors of ten allows a William to know that 12 x 10 is 120 because 12 can be decomposed into its place value components [(10 + 2) x 10]. It will also allow him to reason later around combinations such as 10 x 40. If he knows that ten tens are in one hundred, he needs four times that amount so the answer is 400 [10 x 40 = 10 x 10 x 4]. To reason why 10 x 400 is 4000 – without just counting zeros – is that if ten hundreds (10 x 100) are 1000, four times as many will be 4000 [10 x 400 = 10 x 100 x 4].

With this capacity to reason around *n times as many*, or in this instance *10 times as many,* students need to explicitly reason with the following combinations:

10 x 10	[ten tens]
10 x 100	[ten hundreds]
100 x 10	[one hundred tens]
100 x 100	[one hundred hundreds]
1000 x 1000	[one thousand thousands]

Patterns do emerge. Students will notice them. The urge to find an answer by merely counting zeros is strong. These patterns should be recognized and discussed. That said, it is essential that students develop the reasoning behind the patterns so that they develop the capacity to truly comprehend what it means to have 10 times as much or 10 times less of one quantity than another.

How to Build Such Capacity in the Classroom

What begins in kindergarten with structuring around ten, matters in fifth grade when a robust multiplicative understanding of place value is necessary. The following are places to consider if what is in the published curricular materials at ones disposal is adequate or if clarifications and depth needs to occur.

A. Ten as an organizing unit: Use multistep multiplication and addition and measurement division tasks, beginning in kindergarten to have students explore 10 as a countable and organizing unit and as students progress, to explore the multiplicative relations of place value.

B. Working across as well as within place: Explore more explicitly the questions of how many tens are in the number 783 as well as the more common how many tens are in the tens place. Re-unitizing across places is an important aspect of place value and base ten understanding.

C. Decomposition of number: Breaking numbers into equivalent forms either as addends or factors are key mathematical ideas. Understanding that to calculate 50 x 300 one is not merely subtracting or adding zeros matters. Understanding that to work with the five one needs to decompose 50 into 5 x 10 and 300 into 3 x 100 emphasizes the mathematics of the pattern rather than the surface features of the pattern.

D. Language: Speaking in value rather than in digits is important for very young children. It helps focus on the quantities involved even if those values are large piles of ones in the mind of the child. With older children, speaking in values allows the algebraic properties underlying the operations to be more explicit.

E. Powers of Ten: What are ten tens? (10 x 10). That core fact becomes foundational. Does knowing what are ten hundreds (10 x 100), a hundred hundreds (100 x 100), etc. Reasoning about these combinations and listening to the language each conveys becomes essential in knowing that the product of 67 x 356 is at least above 18,000 as 60 x 300 is the first partial considered.

F. Place Value as a multiplicative relation: From kindergarten to fifth grade, all of the above ideas lead to this single powerful idea: place value is not additive, it is multiplicative in structure. The earlier this concept is nurtured and consistently explored across grade levels the sooner it will solidify. Surface patterns, while observable, should be explored at a mathematical level so that the students move off of the tactical convenience of the surface pattern to conduct procedures to understanding the algebraic principles that govern the operations.

Summary

Place value at its core is a multiplicative relation. It builds slowly over time as children move from organizing their mental models from piles of ones to a multiplicative structure based upon the rate of ten. Some research finds that place value does not solidify until fifth grade (Clark & Kamii, 1996) due to the multiplicative nature of the number system. Prior to that, students' place value understanding is fragile. The selective choice of number combinations set into various problem types, and the use of the language of value around multidigit numbers develops place value concepts sooner in children. Multiplication and measurement division explicitly support this development. Very few published curricula use these operations for this purpose. You, the teacher, need to intentionally infuse the use of these two problem situations into your practice.

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