Extending Place Value & Base Ten Understanding: Building a Multiplicative Rate of Ten

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Extending the conversation

This article extends the conversation begun in *Children's Development of Place Value and Base Ten Understanding: Building a Multiplicative Rate of Ten.* In that piece, the underlying multiplicative structure of place value and the unique properties of multiplication as an operation were explored as a means to understand children's development of the rate of ten. There are some key ideas reviewed here that are pertinent to the extended discussion of this article. They include:

- The concept of the rate of ten
- The capacity to unitize and re-unitize a number
- Decomposition of number in addends and/or factors
- The role of the distributive, commutative and associative properties in multiplication and division
- The unique properties of multiplication as an operation

These mathematical ideas are part of the "knowledge package" needed to have a "profound understanding" (Ma, 1999) of place value as a rate of ten.

Place value is more than naming the value of the digit within a particular location. What is also an aspect of place value is the capacity to comprehend values across places. As an example, consider the number 783. The following expressions are all true:

783 x 1

700 + 80 + 3 *or* 700 x 1 + 80 x 1 + 3 x 1

78 x 10 + 3 x 1

78.3 x 10

7 x 100 + 8 x 10 + 3 x 1

7.83 x 100

Each of these expressions captures important mathematical perspectives as one decomposes and re-unitizes the original quantity. It is the flexibility of reading a number's value across places that is equally important as within place. The combination allows the profound understanding of place value as **a rate of ten** to emerge and solidify within a child.

Place value as the coordination of a ratio, as a multiplicative relation is at the core of this rate of ten concept. For every new ten gathered, a simultaneous accumulation of ten ones also occurs.

Unitizing is the cognitive assignment of a unit of measure to a given quantity. **Re-unitizing** is the decomposition or reconfiguring of a quantity in terms of less or more composite units (Lamon, 1996). This cognitive ability captures this capacity to look at a number and shift the unit reference while holding the original quantity simultaneously. That 3.1 million (three and three-tenths million) is equivalent to 31 hundred thousands and 3,100,000 ones is an example of this cognitive process. This is the mathematical basis for working with mixed decimals such as 2.4 ones (two and four-tenths ones). Thinking of that quantity as 24 tenths (2.4 = 24/10) makes it easier to divide by 4 ones; the result of which would be 6 tenths, 6/10 or .6 of one.

Multiplication as an operation is uniquely different than addition. Multiplication is:

- About making units of units which then can be iterated/repeated
- About the coordination of units among elements
- Unit transforming
- The capacity to scale up and down; seeing multiplication as *n-times as many*

The following sections explore how using the rate of ten and the capacity to re-unitize quantities is reflected in abstract multidigit multiplication and division strategies. A focus on these two mathematical ideas allows for a deeper understanding of how the underlying properties of place value combines to support a stronger foundation for middle school mathematical ideas.

Multidigit Multiplication – Multiplication With Decimals: A Case for the Explicit Use of the Rate of Ten

Researchers have outlined how students who are allowed to develop their own strategies in multiplication (Ambrose, Baek, & Carpenter, 2003) progress in the use of strategies. Early attempts at multiplication emerge out of repeated addition to forms of doubling and complex doubling. The use of the distributive property emerges as students' understanding of the partitioning process into addends grows, e.g. 42 = 40 + 2. The emergence of the distributive property at an additive level can be seen in the work of William. To solve how many eggs are in 10 dozen cartons, William works on paper writing, "10 x 10 = 100; 10 x 2 = 20 so the answer is 120." This second grader is coming to understand that numbers can be broken into combinations of addends, the parts multiplied, then added to find the total. Students begin to understand that the number of combinations to be multiplied increases with the number of partitions.

This section explores how a focus on the relations within and across the places of a number allows students to gain a robust understanding of the base ten system. This conceptual understanding underpins the procedural knowledge necessary for the computational proficiency required in multidigit multiplication and division whether working with multidigit whole numbers or with decimals. A fifth grade Common Core standard requires that students, *Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left. (CCSSI, 2010, 5.NBT.1, p. 35).* To have a sound understanding of this standard, the mathematical ideas previously highlighted converge to form a comprehensive network of related ideas.

Whole Numbers

Consider the following:

$$26 \times 42$$

To solve the problem using algebraic properties, the numbers need to be decomposed either into addends, factors, or a combination of the two. Decomposing numbers into addends is typical of most students' early work and is the mathematical basis for multiplying binomials and the area model/standard multiplication algorithm.

$$(20 + 6) \times (40 + 2)$$

The requisite combinations are formed, multiplied then the partials are added.[1]

$$20 \times 40 + 20 \times 2 + 6 \times 40 + 6 \times 2$$

$$800 + 40 + 240 + 12$$

$$1092$$

Many mathematical curricula treat the partials 20 x 40 and 6 x 40 from a surface pattern only. Called "front-end multiplication," one multiplies 2 x 4 then "add two zeros." This may be an observed visual pattern but it does not explain mathematically why this pattern holds true.

Think, however, as described earlier in the previous article, how students even as young as third grade are able to consider how 20 can be represented as 2 x 10 and 40 as 4 x 10. This decomposition process exposes the place value rate of ten. Thus 20 x 40 is calculated knowing 2 x 4 but 10 x 10 are factors that remain to be considered. Thus 8 x 100 (read as, what are 8 hundreds?) is explicitly considered rather than regarded as the superficial machinations of a surface pattern. Fifth graders considering the Common Core standard of knowing that the next place larger in a multidigit number is 10 times greater is achieved when place value, and its factors of ten, is explicitly understood as a rate of ten.

[1] While multiplication as an operation is commutative, multiplying the larger terms first is privileged as it typically arranges the partials from largest to smallest, albeit not exclusively. This is the order followed when multiplying two binomials: $(2x + a)(4x + b)$ to get $8x^2 + 2xb + 4xa + ab$. The mnemonic term for this order is F.O.I.L. Note that the procedural order when using the traditional algorithm is L.O.I.F. Other combinations, O.L.I.F, I.L.F.O… are all mathematically legitimate as multiplication is commutative. They are just uncommon.

The research perspective on the standard multiplication algorithm taken by this project focuses on the role of language and 'making the mathematics visible' (Brickwedde, unpublished manuscript 2010) to its users. Rather than masking the mathematics through the procedural use of single-digit language, the explicit use of factoring the place value and the commutative and associative aspects of multiplication as an operation strengthen both the conceptual and procedural knowledge of the user.

Decimals

A focus on the rate of ten extends into how multiplication with decimals can be understood. Moving to a place to the right in a multidigit number still involves the rate of ten. However, each place is one-tenth the size of the larger place before it. The earlier example of 2.6 million (mathematically read as two and six-tenths million), the "two" represents the number of whole millions. The '6' represents six-tenths (6 ÷ 10) of one whole million. Using a simpler example, consider the following:

$$4 \times 2.6$$

Many published math curricula instruct students to ignore the decimal point, multiply the numbers as if it was a whole number, then locate the position of the decimal point last. The number of decimal places to have in the product is determined by counting the number of decimal places in the original numbers. But what is the mathematics behind this surface treatment.

The re-unitizing of a number, described previously, is the ability to find an equivalent value of a number but from a different unit perspective. This is a core idea in fraction equivalencies. Consider the number 2.6, read as two and six-tenths. The '2' is two ones and '6' is sixth-tenths of one.

$$2.6 \text{ ones} = 2 \text{ ones} + \frac{6}{10} \text{ ones}$$

The standard practice of "ignoring the decimal" masks the importance of the role of re-unitizing. 2.6 ones does not equal 26 ones, but it does equal 26 tenths.

$$2.6 \textit{ ones} = 26 \textit{ tenths}$$

or

$$\frac{2.6}{1} = \frac{26}{10}$$

The ability to re-represent the decimal as a fraction exposes the underlying mathematical relations, the re-unitized equivalent values, and the rate of ten. Using the distributive property to conduct the calculations, the following makes visible the underlying mathematics.

$$4 \times 2.6 = 4 \times (2 + .6) = 4 \times (2 + \frac{6}{10})$$

$$= 4 \times 2 + 4 \times \frac{6}{10}$$

$$= 8 + \frac{24}{10}$$

$$= 8 + 2 + \frac{4}{10}$$

$$= 10.4 \text{ ones}$$

Looked at another way…

$$4 \times 2.6 = 4 \times 26 \div 10 \textit{ or } \frac{4 \times 26}{10}$$

$$= (4 \times 20 + 4 \times 6) \div 10$$

$$= (80 + 24) \div 10$$

$$= 104 \div 10$$

$$= 10.4 \text{ ones}$$

Here is another example, this one involving multiplication of two decimals.

$$2.4 \text{ x } 5.3 = \frac{24}{10} \text{ x } \frac{53}{10}$$

$$= \frac{24 \text{ x } 53}{10 \text{ x } 10}$$

$$= \frac{20 \text{ x } 50 + 20 \text{ x } 3 + 4 \text{ x } 50 + 4 \text{ x } 3}{10 \text{ x } 10}$$

$$= \frac{1000 + 60 + 200 + 12}{100}$$

$$= \frac{1272}{100}$$

$$= 12.72$$

This is the mathematics behind asking students to “ignore the decimals and pretend they are whole numbers.” By making the mathematics visible, the factors of ten become explicit, thus helping students to reason with the numbers rather than working merely at a procedural level. To do the above with understanding, the following needs to be understood:

- That 2.4 ones is equivalent to 24 tenths
- The distributive property of multiplication over addition
- Order of operations
- To read across place values to know that 1272 hundredths can be re-unitized to be equivalent to 12 and 72 hundredths ones (12.75)

This is all within the range of understanding of fourth and fifth graders if this type of thinking is explicitly cultivated. It is especially possible if the underlying algebraic structures of algorithms are the foundation of student thinking.

Several mathematical ideas merge to form a broader conceptual field: re-unitizing a number into equivalent values (hundredths to ones), re-representation of expressions into equivalent forms (decimals to fractions, fractions to decimals), decomposition of number into addends or factors, the distributive property of multiplication over addition, place value as a rate of ten, and the relationship of what can be done with whole numbers and how it extends to working with decimals. This capacity to re-unitize a number into an equivalent form (2.6 ones = 26 tenths) is directly related to the issue in long division when students are asked to divide to one, two, three, or more decimal places.

Division of decimals

The research of Ambrose, Beak & Carpenter (2003) have shown that children’s initial abstract division strategies evolve from a building up strategy using multiplication. Dutch researchers (van Putten, et. al 2005) have looked at students’ “progressive mathematization,” the increasing capacity to think in efficient quantities and in using scale factor in the use of the partial quotients strategy. It is this later research that is most pertinent to the re-unitizing issues being highlighted in the discussion of decimals in this article.

The partial quotient strategy allows students’ initial estimates and number sense to be captured as they begin to solve a division problem at a numerical level. It is based on talking about the dividend in terms of its value rather than the traditional algorithm’s digit-based approach. Like the traditional algorithm, it works towards the elimination of the dividend by working backwards towards zero. An example of an early use of partial quotients is as follows:

8)254	
−80	10
174	
−80	10
94	
−80	10
14	
−8	1
6	31 R 6

The introductory process begins with the largest combination of “groups of eight” the child knows (a measurement division structure). As the students’ number sense matures, and through coaching by the teacher, questions like, *yes you could do 10, but are you close or far away? Could you double or triple the amount to save yourself some time?* Such prompts scaffold the student’s thinking to use scale factor to double (20 groups of eight to use up 160 units) or triple (30 groups of eight to use up 240 units) the original estimate. The increasing efficiency over time allows students to regard the numerical relations that are visible to them in the number combinations (8 x 3 x 10 = 24 x 10 = 240). This “progressive mathematization” measures the increasing capacity to think in scale.

What if, however, one now wishes to convert the remainder of six to a decimal? How does one divide 6 by 8? Many curricula, even those that have received National Science Foundation (NSF) funding, discuss about “adding a zero” and pretending that the decimal does not exist for the moment. There is a real mathematical idea behind this surface treatment that is worthy of explicit exploration. Instead of asking students to pretend, what if we ask them to grapple with the re-unitizing of the numerical values. Consider the following:

8) 254	
−240	30
14	
− 8	1
6.0	

At this point in the procedure, “6,” as in “six ones” is more useful to be thought of as “60.” The sixty, however, is not “60 ones. Rather it is “60 tenths.” Re-unitizing six into sixty-tenths allows for easier arithmetic. Seven tenths (.7) times eight uses up fifty-six tenths (56/10). That quantity happens to be equivalent to “five and six-tenths” (5.6). It is the toggling back and forth between the unit conversions, while awkward initially for students, develops a deep understanding of equivalencies, unit transformation, and unit coordination; key aspects of being able to think multiplicatively in working with ratios and proportions later on in middle school. Thus the division of the decimals in this example continues as follows:

$$
\begin{array}{r|l}
8\,)\,254.00 & \\
\underline{-240} & 30 \\
14 & \\
\underline{-\ \ 8} & 1 \\
6.0 & \\
\underline{-5.6} & .7 \quad \textit{(8 groups of .7 equals 56 tenths or five and sixth tenths [5.6])} \\
.40 & \\
\underline{.40} & \underline{.05} \quad \textit{(8 groups of five-hundredths equals forty-hundredths)} \\
0 & \\
 & 31.75 \quad \text{The answer is 31.75}
\end{array}
$$

The mathematics just described link directly to the conversation on multiplication with decimals and the power of re-representing decimals as fractions. Consider the same problem written differently:

$$
\begin{aligned}
6 \div 8 &= \frac{60}{10} \div 8 \\
&= \frac{60 \div 8}{10} \\
&= \frac{(56+4.0) \div 8}{10} \\
&= \frac{7.5}{10} \\
&= .75
\end{aligned}
$$

Division with decimals is another topic where surface pattern rules have predominated with users not understanding the mathematics about why the patterns work. Consider 8.1 ÷ 3.6. Presented in the "long division" form of representation, $3.6\,\overline{)8.1}$, students are directed to just "move the decimal points," do the division using the whole numbers, then trust that the answer is correct as no requirement to bring the decimals back is necessary. Why is that? What mathematically is happening that allows one to trust that 8.1 ÷ 3.6 = 81 ÷ 36. Re-representing these two expressions into fractional form, as all fractions are statements of division, allows the underlying mathematics to become visible.

$$8.1 \div 3.6 = 81 \div 36$$

$$\frac{8.1}{3.6} = \frac{81}{36}$$

$$\frac{8.1}{3.6} \text{ x } \frac{10}{10} = \frac{81}{36}$$

The identity property of multiplication or proportional reasoning governs why this pattern works. Transforming the original numbers into equivalent relations is not dissimilar to the act of decomposition and reconfiguring of quantities to make them easier with which to work.

Re-unitizing a number, equivalent forms of representation, understanding place value as a rate of ten, and the language of value combine to form this network of related conceptual ideas and form the basis of having a profound understanding of the operation of multiplication and the structure of the base ten system.

Building Capacity to Use the Rate of Ten

This next section looks at key instructional practices and tasks that can be used with students to build the capacity to reason around the rate of ten.

Language

It has been expressed in the initial article on place value that to understand the underlying mathematics, its decompositions and units of measure, speaking in value is imperative. Watching ones language helps the student cognitively visualize and coordinate the units as decomposition of the numbers and transformations of units occur. With 26 x 42, it is a "twenty" (20) and a "forty" (40) that are multiplied as the first partial rather than a 2 and a 4. However, knowing that 20 = 2 x 10 and 40 = 4 x 10 allows one to multiply 2 x 4 to get 8 but one is mathematically left with 10 x 10 making 100, thus 8 x 100 is 800. Using the language of value allows one to more accurately trace the mathematics. The factors of ten are explicitly exposed and reasoning around place value as a rate of ten is in the foreground of ones work rather than mere memorized surface patterns.

Reading and using the correct mathematical language for the expression 2.4 million is a harder adjustment for adults. The mathematically correct way to read that is "two and four-tenths million" not "two point four."[2] There are times when just merely reading off the digits to someone who needs it recorded may make sense but the common street habit of using "point" language may inadvertently inhibit place value understanding. It is important that students know what .4 million (four tenths of a million) is worth.

In helping to build capacity to verbalize the values of numbers as well as to build fluency in working around landmarks of ten, a mathematical warm-up task used with whole numbers can be implemented using decimals. Example, in as few "jumps" as possible,

You are at 36, how much to get to 100?

$$36 \xrightarrow{+4} 40 \xrightarrow{+60} 100, \text{ the answer is 64.}$$

You are at 256, how much to get to 1000?

$$256 \xrightarrow{+700} 956 \xrightarrow{+44} 1000, \text{ the answer is 744.}$$

You are at 7.14, how much to get to 8?

$$7.14 \xrightarrow{+.06} 7.20 \xrightarrow{+.80} 8, \text{ the answer is .86}$$

You are at 50, go back .18

$$49.82 \xleftarrow{-.08} 49.90 \xleftarrow{-.10} 50$$

You are at 70, go back 2.14.

$$67.86 \xleftarrow{-14} 68 \xleftarrow{-2} 70$$

[2] There is a mathematically correct context for the "point" language. 5.2 can be the numerical code for a sorted list as in "unit five, lesson 2." In standards-based documents 5.2.1 could stand for grade five, standard two, benchmark one." A "point 4 million" does not convey the same mathematical relationship as "four-tenths of a million."

Doing this task with decimals emphasizes the language of values. Adding "six-hundredths" to get to "seven and twenty-hundredths," then eighty-hundredths" or "eight-tenths" to get to "eight" carries more meaning than adding "point oh six" to get "seven point two." This is more than a pedantic issue of language snobbery. Expecting students to talk in values elevates the mathematical understanding within the learning environment. "Four-tenths of a million" carries more meaning than "point four million." The place value and the rate of ten is more explicit.[3]

Re-unitizing Number & Following the Units

What is four-tenths of a million? If we return to the example of 2.4 million (two and four-tenths million), we know that the "two" represents two groups of one million. The "four tenths" of the million is equivalent to four hundred thousand or 400,000.

In mathematics the unit of "ones" is not reflected typically when numbers are verbally described. Nevertheless, there are indeed 400,000 ones in four hundred thousand. Asking students to consider the ones as a unit of measure becomes useful in helping students to decompose and reconfigure numbers into equivalent forms. Consider the following series:

1.23 thousands ←→ _________ ones

.02 millions ←→ ________ tens

______ tens ←→ 2,652 hundreds

Which is greater or are they equal? (<, >, =)

.02 tens 2 ones

.35 million 45 hundred thousand

Developing the Distributive Property

The distributive property is the basis of the multiplication algorithms most frequently used. An underlying conceptual skill is the decomposition of number into equivalent sub-units. Thus 26 x 42 is typically decomposed into 20 + 6 x 40 + 2. The same conceptual idea can be extended to multiplying decimals as in 2.6 x 4.2. These numbers can also be decomposed into 2 + .6 x 4 + .2, followed by the multiplication of the four partial products.

Using true/false and open-ended number sentences (Carpenter, Franke, & Levi, 2003), instructional tasks can be developed that help students reason around the distributive property. Consider the following tasks:

1.2 x .6 = 1 x 6 + 2 x 6 T/F?

4 x 1.2 = 4 x 1 + 4 x .2 T/F?

4.2 x 1.2 = 4 x 1 + .2 x .2 T/F?

.2 x 1.2 = .2 x 1 + ___ x .2

3.1 x 4.4 = 3 x 4.4 + .1 x ____

Reasoning With the Rate of Ten

Knowing that 10 x 10 (ten tens) are 100 is useful, as is knowing 10 x 100 (ten hundreds or one thousand). But what of the following?

What is…

.1 of 10	(one-tenth of ten)
.1 of 100	(one-tenth of one hundred)
.1 of .1	(one-tenth of one-tenth)
.1 of .01	(one-tenth of one-hundredth)

Other ways of reasoning around the rate of ten is to become comfortable factoring the tens underlying the place value of the numbers involved.

5 x 2.6 = 5 x 26 x .1 or $\frac{5 \times 26}{10}$

This act of factoring elevates the use of the associative property of multiplication.

.2 x .02 = 2 x 2 x .1 x .1 T/F?

4.2 x .6 = (4 x 6 + 2 x 6) x ___ x .1

4.05 x 3 = 405 x 3 x ___

Another type of instructional task to help reason around the rate of ten is to focus on the relational thinking when comparing a known quantity with a similar quantity.

It is true that 174 ÷ 8 = 21.75. Use estimation and number sense to determine where to place the decimal point in each problem below. Explain why you placed the decimal point where you did.

17.4 ÷ 8 = 2 1 7 5

1740 ÷ 8 = 2 1 7 5

1740 ÷ 80 = 2 1 7 5

Using Multistep Multiplication and Measurement Division Problems

The work of Empson & Levi (2011) and Carpenter, Femmema, Franke, Levi & Empson (1999) explore the role how multiplication and measurement division can be used to support the development of place value and base ten understanding. The following are examples with whole numbers.

There are 7 boxes of markers on the store rack. Each box has 10 markers in it. How many individual markers are in all of the boxes?

The art teacher has 243 markers in baskets for students to use for a project. She asks a student to place the markers back into boxes. If 10 markers fill up at box, how many full boxes can the student fill?

[3] It is also legitimate to help students recognize that the public (news anchors, adults in their world) will read mathematical values such as 3.2 million as "three point two million." The standard language in math and science classrooms needs to be of value to develop and enhance the number sense and concepts of measure.

Each of these tasks, with the use of ten as the organizing quantity, guides the student to consider the multiplicative relation of the base ten system. 70 equals 7 x 10. 243 has 24 whole groups of ten within the number. The same two problem types can be selected to have students explore the base ten relations through the use of decimals.

> *It takes .5 of a yard of fabric to make a pillow. How much fabric is needed to make 12 pillows?*
>
> *To make a batch of chocolate chip cookies, the recipe calls for .1 of a tablespoon of baking powder. How many batches of cookies can be made if there are 4 tablespoons left?*
>
> *It takes a .1 of meter of Velcro to use as on a child's shoe at the factory. If a machine only has 5.2 meters of Velcro left on the spool, how many more shoes can be finished before the supply of Velcro runs out?*

The context of the different scenarios allows students to visualize what is happening allowing them to construct meaningful strategies to solve for the unknown information. The public sharing that would accompany the strategies allows students to compare, contrast, and contemplate efficiencies. The capacity to reason with the relationships strengthens over time.

Summary

Language, decomposition of number, derived and relational thinking, equality, unit coordination and re-unitizing, and keeping the mathematics visible are all elements interwoven to create a conceptual field needed to comprehend and work with a multiplicative rate of ten. This is the core of the base ten system. It is the foundation upon which rational number and proportionality are built in middle school mathematics classrooms. It is a house of bricks rather than of straw or sticks. Building fluency and understanding takes time and consistent and persistent conversations over several grades. The outcomes, however, are highly beneficial for developing future mathematical concepts.

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