Students' Multidigit Addition & Subtraction Strategies

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Students, when left alone, have been found to naturally begin adding multidigit numbers from the larger part of the number to the smaller (Kamii, 2000). This is largely due to following the language of numbers. If I want to add 54 and 37, I hear or say fifty and thirty before I hear or say four and seven. This concept of working with the larger part of the number is supported by our own adult mental math strategies when we are out and about in the grocery store or restaurant. Our tendency is to group the tens, round off numbers to make them quickly usable, and look for familiar landmark combinations in order to generate easier chunks of numbers with which to work. These mental math strategies, as we call them, emerge early on in children who have not been exposed to the traditional, school algorithms of starting with the ones, then combining or separating the tens. What researchers found, is that students who use these natural, *invented* strategies, develop stronger number sense and place value concepts earlier than students who use the traditional algorithms (Carpenter, et. al. 1998, Fuson, et. al. 1997, Hiebert, ,et. al. 1997).

What Are These Invented Strategies - Addition

Some educators and parents hear the term "invented strategies" and interpret it to mean "anything goes." However, once one looks at the mathematical options that a child considers, the strategies fall within three (addition) or five (subtraction) distinct categories. Consider the following addition problem:

$$\begin{array}{r} 54 \\ \underline{+37} \end{array}$$

Remembering that children follow the language of the numbers, one typical strategy is to add the tens, then add the ones. Thus a child would say, "50 + 30 = 80; 4 + 7 = 11; 11 = 10 + 1 so 80 + 10 + 1 = 91". Vertically, the child's work would look like the following:

$$\begin{array}{r} 54 \\ \underline{+37} \\ 80 \\ \underline{11} \\ 91 \end{array}$$

This strategy is known as the ***Tens & Ones Strategy***[1]. There are other options, though. In the Tens & Ones strategy, both numbers are partitioned into their place value components, the tens are added, then the ones are added. If another ten is created, that is combined with the tens, then the remaining ones combined to solve for the final number. Consider the difference between that approach towards combining the numbers with this next solution strategy:

$$\begin{array}{r} 54 \\ \underline{+37} \end{array}$$

Verbally, a child says,

54 + 30 → 84 + 6 → 90 + 1 → 91

A variation of this would look like as follows:

50 + 30 → 80 + 4 → 84 + 6 → 90 + 1 → 91

In each of these two solutions, the numbers are worked with *incrementally*. In the first solution, only one of the two numbers was partitioned into its place value components, followed by incrementally adding the tens then the ones. In adding on the ones, advanced users of this strategy decompose the remaining single digit number into a combination to make a ten (4 = 3 + 1; 7 + 3 = 10), then join the unused portion to the newly made decade number, in this case, the 90 (90 + 1 = 91).

In the latter solution, both numbers were partitioned but, unlike the tens and ones strategy, the numbers are combined incrementally: the tens (50 + 30) then sequentially the 4, then 6 and finally the 1 (7 = 6 + 1). These two solutions are variations of what is known as the ***Incremental Strategy***.

Given the particular numbers in the example that is being used here, there is a third option that students may use. This strategy is less frequent than the previous two, and usually develops later in that it is triggered by numbers within the problem. Observe:

$$\begin{array}{r} 54 \\ \underline{+37} \end{array}$$

Verbally it sounds as follows:

37 is close to 40 so…

54 + 40 → 94 – 3 → 91

This solution is called the ***Compensation Strategy***. The child adjusts the one number by adding 3 to the 37 to make it easier to combine to the second multidigit number. Once accomplished, the child needs to readjust the new total in order to find the final answer.

[1] The original researchers (Carpenter, et al., 1999) are shifting to the label ***Combining Like Units*** in order to capture the commutative property aspects of arriving at the same answer whether or not one starts adding the tens first then the ones or adding the ones first then the tens. The commonality is that *like units* are combined first. It also captures better when one is adding numbers in the hundreds or thousands. Combining Like Units is more inclusive a label.

A variation of this compensation strategy is based upon decomposing the 54 into 51 + 3 and re-associating the 3 with the 37. Mathematically this looks like the following:

$$\begin{aligned} 54 + 37 &= (51 + 3) + 37 \\ &= 51 + (3 + 37) \\ &= 51 + 40 \\ &= 91 \end{aligned}$$

A student need not write out the mathematics in such detail. These steps are typically done mentally and efficiently.

One strength of each of the above strategies is that, by starting with the larger part of the number first, the child immediately has a 'ballpark' sense of the size of the final answer. If one starts with 50 + 30 or 54 + 30, the reasonableness that the answer is going to be at least larger than 80 or 84 is quickly established. Another strength is that because the child is describing the numbers in terms of the value, meaning the 50 is 50 rather than a 5, the values of the numbers are being worked with rather than merely working with a string of single digits.

Mathematical Concepts Underlying the Strategies

Each of the above solution strategies draws upon number concepts and algebraic properties that a child needs to grasp in order to use each with understanding. Key concepts and properties follow:

Decomposing & Partitioning Numbers[2]*:*

Decomposing Numbers: Decomposition is based on the concept that numbers are composed of subsets of other numbers; that 7 is 6 + 1, 5 + 2, and so on. In the context of 54 + 37, being able to decompose 7 into 6 + 1 in order to make a new ten (54 + 6 = 90) allows the child to eloquently navigate the number system to find a quick solution rather than counting on the seven one by one. Similarly, decomposing 54 into 51 + 3 in order to re-associate the 3 with 37 to make 40 demonstrates how the properties of addition function. Fluency with the concept of decomposition of number allows students to move away from the inefficient dependency of counting on or back by ones.

Partitioning Numbers: Partitioning refers to the specific decomposition of a number into its place value components. In this example, 54 is partitioned into 50 and 4. This is a concept that begins to emerge among first and second graders. The concept is based upon more than just naming the digit's value according to its location. Knowing that 50 is five tens (5 x 10), a multiplicative relationship, is an ultimate understanding that takes time to mature. Some research indicates that a robust understanding of place value, meaning place value as a multiplicative relation, does not fully mature until fifth grade (Clark & Kamii, 1996). Prior to this time, a child's place value understanding remains fragile (Kamii & Dominck, 1998).

[2] Partitioning is a subset of decomposition of number. Partitioning will be used to denote the place value components, while decomposition will be used to denote non-place value components such as 7 = 6 + 1 or 54 = 51 + 3.

Dice Combinations & Facts to 10: When faced with adding 84 + 7, to know to decompose the 7 into 6 + 1, as opposed to 4 + 3, is based upon an immediate recognition that 4 + 6 makes ten. Instantly judging the 84 and the 7 in this context is based on fluently knowing the fact family of 7 (the dice combinations) as well as the combinations of 10. Fluency with these combinations allows strategic judgments to be made leading to greater efficiency.

Zero: Knowing that 50 and 4 makes 54 is partially based upon the knowledge that zero plus a number gives you that number ($0 + a = a$).

Incrementing: This skill arises out of the ability to skip count beginning from any number. Using the base ten system, incrementing by tens from any number becomes an important skill to utilize when considering the distance between two numbers and for combining larger numbers.

Compensation: The child needs to comprehend the impact of the adjustments he or she makes to the numbers with which one is working. In the case of this addition example, if 37 is increased by 3, a readjustment of 3 must be made to find the solution to the original problem.

54 + 37 ≠ 54 + 40, however
54 + 37 = 54 + 40 – 3 as 37 = 40 – 3

Equality: Inherent in both decomposing and reconfiguring numbers and in compensation is the relation of equality.

54 + 37 ≠ 54 + 40 but…
54 + 37 = 51 + 3 + 37, or
54 + 37 = 54 + 40 – 3
7 = 6 + 1 = 5 + 2 = 4 + 3

As numbers are partitioned, decomposed, and reconfigured, a child needs to maintain that the number sets are *conserved* to create equivalent sets. Many children view the equal sign (=) as an operation symbol, that the "answer comes next." This is a misconception. The equal sign is a relational symbol, one of comparing equal values.

The Standard Algorithm for Addition[3]

The *Standard Addition Algorithm*, as it is classically taught in the United States, presents some complexities for many students. Consider the language used in classrooms to solve the previous problem:

$$\begin{array}{r} {}^{1}\\ 54 \\ \underline{+\,37} \\ 91 \end{array}$$

"4 + 7 is 11, put down the 1, carry the 1, 1 + 5 + 3 = 9, the answer is 91."

Young children are still very literal thinkers. They typically can only consider one attribute of an object at a time. Their

[3] The Common Core State Standards for Mathematics (CCSSI, 2010) does not require students to engage with the Standard Algorithm in addition and subtraction until 4th grade. In grades 1 through 4 students are expected to add and subtract *algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.* (2.NBT.5, 3. NBT.2)

ability to consider that a multidigit number is both a collection of ones as well as a combination of relationships organized into tens and ones is fragile while under development. The use of single digit language when working with multidigit numbers raises the cognitive difficulty of what is essentially a *ones and tens* strategy. The values of the numbers are lost. The more emergent child literally thinks that the 5 in fifty has only a value of 5. In order to use this single-digit language with understanding, a child needs to conserve, that while saying 1 + 5 + 3 , that that is the same as 10 + 50 + 30, and, while splitting the 11 into a 1 below and a "1" above, that the "1" is really a ten and that the two "1s" are still the equivalent of 11. From a child's developmental status, this is a significant strain on his or her cognitive abilities. This is why many children who are taught this procedure have significant place value problems even though they can use the procedure to get a correct answer (Fuson, 1990, Fuson, et al., 1997, Kamii & Dominick, 1998).

This need not happen, however. *Simply changing the language that is used while working with this strategy* by itself will improve a child's understanding of how they are combining the numbers. Keeping all numbers visually whole rather than splitting them apart into single digits also reduces the amount of cognitive tracking a child needs to maintain. Consider the problem once again:

$$\begin{array}{r} 54 \\ \underline{+\ 37} \\ 11 \\ \underline{80} \\ 91 \end{array}$$

"4 + 7 = 11; 50 + 30 = 80; 80 + 11 = 91."

The language reflects the value of the numbers rather than merely the digit's name. Visually, the 11 and the 80 appear as whole numbers and reflect the full value of the combined sets. While the child may not have a full idea until the second step that the answer will be above 80, as compared to the earlier strategies listed, the child is able to maintain a clear number sense while working through the problem. If the child uses this *Ones & Tens Strategy* (11 + 80) where another child in the same classroom uses the *Tens & Ones Strategy* (80 + 11), the whole class is able to explore the algebraic principle of commutativity. It doesn't matter if you start with the smaller or the larger. The answer remains the same. The choice of starting with the larger or smaller then becomes an issue of strategy. Are you closer to your answer if you start from the larger (80) or with the smaller (11) and how the ability to conduct these calculations mentally may be easier starting with one more than the other.

Multidigit Subtraction Strategies

Let us consider the same two numbers, 54 and 37, but now in a subtraction context.

Tens & Ones Strategy:

$$\begin{array}{r} 54 \\ \underline{-\ 37} \end{array}$$

The child starts and says" "50 – 30 = 20." Stop! Now, a new mathematical issue needs to be considered. Depending upon how solid the child's understanding is of the number system, two options emerge if the child is to remain in the Tens & Ones mode. By custom, we were taught to say, *"I can't take 7 from 4.* This is not mathematically true. One mathematical option is to say,

"4 – 7 = $^{-}$3 , so 20 – 3 = 17."[4]

Another mathematical option typically heard is,

"I can't take 7 from 4; I'm short 3, so 20 – 3 = 17."

In the former case, the child grapples with the idea that zero is a countable unit on the number line and that negative integers exist on the other side. In the latter, the difference between these two numbers is understood, then knowing whether of not that difference is something that needs to be subtracted from the remaining tens.

When children first attempt this strategy, they typically lose track of which direction they need to continue in. Many times they will understand that the difference between 4 and 7 is 3 but they add the 3 on to the 20 to get an answer of 23. Finding ways to talk through this conundrum with students in the classroom, by modeling the situation with tools or pictures, allow students to bridge their understanding to a correct conception of the problem. This strategy clearly draws out the mathematical concept of *zero.* Zero is no longer defined as just "the empty set", as in "zero is nothing," but rather zero is a countable unit on the number line. Zero in this context is the reflective point between the domains of positive and negative integers. The fact that zero can be bridged and that negative integers can be integrated into the working knowledge of whole numbers is a powerful mathematical idea. However, if a student is uncomfortable going in that direction, simply being able to articulate that one is "short three" and know that that three has to come out of the remaining quantity of tens develops a mathematical skill that is equally legitimate and powerful. The Tens & Ones strategy remains accessible to both types of learners.

Incremental Strategy:

$$\begin{array}{r} 54 \\ \underline{-\ 37} \end{array}$$

"54 – 30 → 24 – 4 → 20 – 3 → 17"

A variation is,

"50 – 30 → 20 + 4 → 24 – 4 → 20 – 3 → 17"

Yet another is,

"50 – 30 → 20 – 7 → 13 + 4 → 17"

In each case, the student needs to track the implication of whether the incremental unit with which they are working needs to be subtracted or added as the number system is being negotiated.

[4] Note: there are mathematical issues with children saying $^{-}$3 and then proceeding to write 20 – 3 versus 20 + $^{-}$3. The instructional emphasis at this point in the learning is typically placed on allowing the child to follow his or her intuitive lead and leave to a later time to explore the nuances of why 20 + $^{-}$3 = 20 – 3.

Compensation Strategy:

$$\begin{array}{r} 54 \\ -\,37 \\ \hline \end{array}$$

"54 – 40 → 14 + 3 → 17."

A variation is,

"54 – 37 = (54 + 3) – (37 + 3) = 57 – 40"

This strategy also generates many false starts as a child attempts to initially use it. With this first version, since 3 is added to the 37 (in addition whatever you add you then subtract) here the child is likely to attempt to subtract an additional 3 from the 13, thus getting 10 as the answer. What a child needs to consider, is that by changing the 37 to 40, 3 too many were taken away from the set so 3 more needs to be added back on. Many children who first attempt this strategy give up and abandon it. It is very worthwhile to generate classroom discussions to demonstrate the underlying mathematics of the strategy as it will add to the depth of the knowledge of how the number system works. The concepts of relational thinking, equality, and maintaining difference between equivalent expressions are developed.

American Standard Subtraction Algorithm:

$$\begin{array}{r} 54 \\ -\,37 \\ \hline \end{array}$$

Using language that describes the value of the numbers rather than as a string of single digits, as well as adjusting the format slightly, is one way a child can understand the underlying mathematics of this strategy. Consider the following:

There is not enough to take 7 from 4 so:

$$\begin{array}{rcl} 54 &=& 40 + 14 \\ -\,37 &=& 30 + 7 \\ \hline 17 &=& 10 + 7 \end{array}$$

The mathematical emphasis is on decomposing 54 into an equivalent combination that allows the top number, the minuend, to be larger than the bottom number. Writing the reconfigured numbers to the side, as oppose to the "crossing out" and "borrowing and carrying" notations we are more familiar with, allows the developing child to see clearly the equivalent relations of the numbers. The child's number sense is more readily maintained. The clear mathematical language being used expresses the value of the numbers with which one is working and supports the individual's understanding of how the strategy functions.

The short cut language that is traditionally used, and the short cut notations used to save space, masks the underlying mathematics. It is not that students shouldn't know how to use the short cut version, but taught too soon without the language and visual supports leads to the errors and misunderstandings students have in doing subtraction. Making the mathematics more transparent and using language that tracks the values of the numbers being decomposed raises the level of mathematical understanding and decreases the errors students typically make.[5]

The Difference Between Two Numbers[6]*:*

$$\begin{array}{r} 54 \\ -\,37 \\ \hline \end{array}$$

The first version of this strategy is based upon the knowledge that addition is the inverse operation to subtraction and that subtraction is not only conceptualized by "taking away" something but also as the measure of distance between two points. Therefore,

$$54 - 37 = x$$

…is restructured by the individual into the equivalent representation of…

$$37 + x = 54$$

…or even…

$$54 - x = 37$$

Solving the problem through addition looks like the follow:

$$37 \xrightarrow{+3} 40 \xrightarrow{+14} 54;\ 14 + 3 = 17;\ \text{The answer is 17.}$$

Why so many variations in subtraction?

The strategies for multidigit addition basically fall within three categories: Tens & Ones, Incremental, and Compensation. There are some variations within those three but the options are reasonably concise. Subtraction, on the other hand, falls in five categories, with many variations within each. What is it about subtraction that opens up more possibilities? As a classroom teacher, why is it worth the instructional time to allow students to explore the possibilities? Table I at the end of this article summarizes the different mathematical ideas that are explored by using the various strategies in multidigit addition and subtraction. What is evident reviewing the mathematical ideas needed to master the various strategies is how broad-based an individual's conceptual understanding and capacity for adaptive reasoning becomes knowing more than one single strategy.

Efficiency, Fluency & Flexibility

There is an artificial debate that sometimes emerges over whether or not a child *needs* to know the standard algorithm versus the Tens & Ones, Incremental, Compensation or Difference Between strategies. That debate is one more of politics than mathematics. The mathematical issue is: can the child, with understanding, navigate the number system to combine and separate multidigit numbers with the least amount of effort. That is what efficiency is. However, what is efficient in one context may not be in another. Therefore, knowing a flexible range of strategies is imperative. The child should have a fluid ability to break apart numbers in various combinations and conserve that the recombined sets are of equivalent value. That is where the instructional emphasis should lie, not whether one strategy should take precedence over the other. This notion of efficiency and flexibility be-

[5] Remember that the CCSSM does not introduce subtraction using the American Standard Algorithm until 4th grade. Prior to this subtraction strategies *based on place value, properties of operations, and/or the relationship between addition and subtraction* (2.NBT.5, 3. NBT.2) are taught. The strategies noted in this article all follow the place value language and draw explicit use of various properties of operations.

[6] Some refer to this strategy as the Inverse Operation, but that is only descriptive of 37 + x = 54. It does not describe the measure of distance when using 54 – x = 37. Thus the *Difference Between* label is used here.

comes a clearer issue when a child goes to subtract multidigit numbers.

The addition standard algorithm with understanding is the Tens & Ones strategy except that it starts from the ones place. The standard subtraction algorithm emphasizes decomposing the "top number" (the minuend) where the Incremental strategy focuses on decomposing the bottom number. With either strategy, the ability to decompose numbers is essential. When numbers are close together, as in 42 – 38, the Difference Between strategy is far less work than the Standard Subtraction Algorithm. When subtracting from an even decade or hundred, as in 300 – 27, the Incremental strategy is very efficient, e.g., 300 – 20 – 7, or even Compensation, 300 – 30 + 3. Being flexible with more than one strategy allows each student to make judgments about the given numbers within the particular context, therefore which strategy would be the least amount of work with which to solve the problem. That is the basis of efficiency, making judgments about which strategy to use given the context at hand.

The National Research Council (NRC), (2001) defines *mathematical proficiency* as based upon five interwoven components:

> *Conceptual understanding* – *c*omprehension of mathematical concepts, operations, and relations
>
> *Procedural fluency* – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
>
> *Strategic competence* – ability to formulate, represent, and solve mathematical problems
>
> *Adaptive reasoning* – capacity for logical thought, reflection, explanation, and justification
>
> *Productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (NRC, 2001, p. 116).

Being flexible, efficient, and fluent with a range of multidigit strategies allows such mathematical proficiency to mature and underlying algebraic properties explored.

Summary: Teaching for Understanding

Children need to understand how to navigate the base-ten number system. The properties of number, the concept of equality, and the interrelationships among the operations need to be understood. What also needs to be remembered is that children's logic is different than that of adults. The language that they use to describe numbers is exactly what their brains are configuring. Their ability to hold two attributes at a time is only forming and is insecure for a long time. As teachers and parents, we must attend to the language we use in order to help students make the connections they need to develop a clear and deep understanding of the base-ten system.

Instructional Implications for the Classroom

Teaching Addition and Subtraction Together - Typically, most text series teach addition then subtraction. In this sequence, children have been found to create misconceptions about working with number that they then over generalize to other operations. By teaching addition and subtraction together, children check misconceptions earlier and build a more complete sense of how the number system works.

Add and Subtract with a full range of numbers - Children should work with all types of numbers while solving problems. They should work with single digit numbers at the same time that they work with multidigit numbers. They should work with multidigit numbers that require regrouping and that do not. If the topic of negative numbers arises, let children explore the possibilities. By working with the full range of number combinations students develop a more comprehensive understanding of the number system.

Work with Multidigit Numbers to Develop Number Sense and Base Ten Understanding - Even first graders can solve problems that are the size that have been used as examples in this article, especially if those numbers are attached to a context with which children can readily identify. They may only count everything by ones or direct model with base ten tools, but that is okay. Counting 54 by ones, while tedious to us, helps to develop important mathematical ideas in a child. As numbers get bigger, children will recognize the tedium and seek out efficiencies. If children always work with comfortable numbers, why learn a more efficient strategy?

Try these in your classroom

Do your students know that zero plus a number gives you the number you just added ($0 + a = a$)? Do they understand the concept of zero in terms of how multidigit numbers are constructed (zero as placeholder)? Try these following situations with your students. Observe carefully how they solve the problems. Do they get the answer automatically, or do they have to calculate? Do they give an automatic answer in one situation but have to calculate in the other? What does this tell you about the depth of their place value knowledge?

- Sam has 20 rocks in his rock collection. His mom gives him 3 more rocks. How many rocks does he have in his collection now? [What if he had 100 rocks and his mom gave him 5, 12, or 20 more?]
- Lucinda had 18 rocks in her bag. She took 8 rocks out and put them on her bed. How many rocks does she still have in the bag? [What if she had 113 rocks and 10 were taken out of the bag?]
- Jorge has 20 rocks. How many more to have 24?
- Alisa has 24 smooth rocks and 34 round rocks. How many more round rocks does she have than smooth? [What if she had 10 and 14? 94 and 114?]

Do you want students to explore incrementing by ten from any number? Explore these tasks and notice the particular numbers selected for each.

- Peter has 26 rocks. How many more does he need to have 46 rocks?
- Margaret has 82 rocks in a box. She takes out 50 and sets them on a table. How many are still in the box?
- Linus has 26 granite rocks and 46 quartz rocks. How many more quartz rocks does he have than granite?

In all of these problems, the *structure* on the problem and the *numbers selected* to be in the problem are what gets the child to explore the relationships within the number system. The role of the teacher is to craft *which problems* with *which number combinations* will draw out the *mathematical thinking.* Try it and observe the range of responses among your students. You will be able to discern a great deal about your students place value and base ten understanding.

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Postscript:

The Role of Floating Digits and Forms of Representation in Multidigit Addition & Subtraction Strategies

The floating digits used in any strategy are really there to help free up the working memory so that one can move on to the next part without having to hold onto all the interim parts. This becomes more necessary with the classic language used in the American Standard Algorithm where single-digit language is used instead of working with the value of the numbers. The other reason to use such floating digits is to *save space* so that multiple lines of computation do not have to be used. An argument in favor of the American Standard Algorithm for either addition or subtraction is that each is easily repeatable and elegant in its use of space.

A counter argument is for the student to "watch ones language," meaning maintain the value of the numbers at all times while one is operating on the quantities involved. This makes it easier to hold more in the working memory thus allowing for working with running totals. Building the capacity to hold more in working memory is part of the level of fluency and efficiency that comes over time when working with any strategy with understanding.

Figure 1 shows the typical placement and use of the floating digits in the American Standard Algorithm. The idea of using floating digits as a means to free up the working memory, however, is not unique to the American Standard Algorithm. The same thing can be done with all others. The classic mental verbal script where single-digit language is used sounds like, *I can't take seven from four, so I borrow one from the five, make it a four, carry the one, fourteen minus seven is seven, four minus three is one. The answer is seventeen.*

American Standard Algorithm

$$\begin{array}{r} 4\phantom{\ {}^{1}4} \\ \cancel{5}\ {}^{1}4 \\ -\,3\ \ 7 \\ \hline 1\ \ 7 \end{array}$$

Figure 1

Figure 2 shows how Partial Differences can be used in an equally efficient use of space. Partial Differences requires the use of the language of values and is more readily implemented (although it doesn't have to) by starting with the larger. Another advantage is that Partial Differences is a subtraction strategy students need to execute when subtracting polynomials in upper level mathematics. As you look at *Figure* 2 and follow the mental verbal script that would accompany the subtraction sequence, it would sound like the following: *50 minus 30 is 20, but looking ahead I see 4 -7 will be 'short three' (or negative three) so 20 minus 3 is seventeen.* The goal of the floating digit is to be able to collapse the number of lines by assisting the working memory. The visual assist of the floating digit supports the transition if necessary. However, if the working memory is strong enough, the digits are not required and don't need to be present. The phrase, *I did it in my head,* rings perfectly true. The language of values is maintained internally increasing the conceptual understanding of the mathematics.

Partial Differences

$$\begin{array}{r} 54 \\ -{}_{2}37 \\ \hline 17 \end{array}$$

Figure 2

Floating Digits as Social Conventions

The important thing to remember is that the floating digits are social conventions; they are customs. The placement of these floating memory devices is arbitrary. There is no mathematical rule or property that is violated if the placement of the floating digits is altered. *Figure 3* provides an example. Using the addition standard algorithm, custom typically dictates that the digits showing the ten separated from the eleven is placed on top. But it doesn't have to be. It can just a easily be placed on the bottom and be just as mathematically correct.

American Standard Algorithm

$$\begin{array}{r} 1 \\ 54 \\ +\ 37 \\ \hline 1 \end{array}$$

Figure 3(A)

$$\begin{array}{r} 54 \\ +\ {}_{1}37 \\ \hline 1 \end{array}$$

Figure 3(B)

To Collapse or Not To Collapse

As students are first understanding the strategies it is important the mathematics stay visible so that the structure of the strategy is more clearly monitored. But as numbers get bigger and more complicated, there is something to say about

Difference Between

$$\begin{array}{rl} 54 & {}^{+14} \\ -\ 37 & {}^{+3} \\ \hline 17 & \end{array}$$

Figure 4

the elegance of cleanly displayed mathematics. If a student wishes to use the Difference Between strategy, yet the problem is written in vertical form, how does one notate what mathematical steps are taken but not have to completely rewrite the original problem? *Figure 4* demonstrates a means to do so. It is not conventional but it saves space and time and is elegant in presentation. The mental script would be as follows: *37 plus 3 gets me to 40, plus 14 gets me to 54. 17 is the answer.* This script exactly follows what cashiers say to customers when change is counted out. *Figure 4* is a written format to capture and hold the working memory steps needed to monitor.

As ones mental capacity builds to use Tens & Ones the "80" and the "11" can be held in the head and 91 can be the only number written. As a teacher, you may not visually see what strategy the student used, but accuracy of answer is clearly observable.

Tens & Ones

$$\begin{array}{r} 54 \\ \underline{+\ 37} \\ 91 \end{array}$$

Figure 5

Table One

Key Mathematical Ideas Developed Within Various Multidigit Addition and Subtraction Strategies

Strategy	Addition	54 + 37	Subtraction	54 – 37
Tens & Ones/ Ones & Tens (Standard Algorithm in Addition) Also known as: Partial Sums/Differences Show All Totals (Addition)	• Commutative property • Partitioning numbers into place value components	54 + 37 = (50 + 4) + (30 + 7) = (50 + 30) + (4 + 7) = 80 + 11 54 + 37 = (4 + 50) + (7 + 30) = (4 + 7) + (50 + 30) = 11 + 80	• Partitioning numbers into place value components • Zero as a countable unit on a number line • Negative numbers • “What am I short?” (Debit & Credit)	54 – 37 = (50 + 4) – (30 + 7) = (50 – 30) + (4 – 7) = 20 – 3 = 17
American Standard Subtraction Algorithm (Variation of Ones & Tens) Also known as: Trades First	___	___	• Decomposing the minuend using ten • Partitioning numbers into place value components • Subtracting by decomposing a ten or… • Recall of facts	54 – 37 = (40 + 14) – (30 + 7) = (40 – 30) + (14 – 7)[7] = 10 + 7 = 17
Incremental Strategy Also known as: Counting On/Back	• Decomposition of a number to make a new ten or hundred… • Incrementing by ten(s) from any number (forwards) • Partitioning numbers into place value components • Commutative property • Associative property	54 + 37 = 54 + 30 + (6 + 1) = (84 + 6) + 1 = 90 + 1 = 91 54 + 37 = 50 + 4 + 30 + 7 = (50 + 30) + 7 + 4 = (80 + 7) + (3 + 1) = (87 + 3) + 1 = 90 + 1 = 91	• Incrementing by tens from any number (backwards) • Decomposition of number to get back to an even 10 decade and to go under a decade • Ten facts in reverse, especially a ten decade minus a single digit number • Order of operations • zero property: a number minus itself	54 – 37 = 54 – (30 + 7) = 54 – 30 – (4 + 3) = (24 – 4) – 3 = 20 – 3 = 17 54 – 37 = 54 – (34 + 3) = 20 – 3 = 17 54 – 37 = (50 + 4) – (30 + 7) = (50 – 30) – 7 + 4 = (20 – 7) + 4 = 13 + 4 = 17 54 – 37 = (50 + 4) – (30 + 7) = (50 – 30) + 4 – 7 = (20 + 4) – (4 + 3) = (24 – 4) – 3 = 20 – 3 = 17
Compensation	• Rounding up to the nearest ten, hundred… with adjustment at end to re-establish equilibrium • Equality • Decomposition of number to then re-associate • Associative property	54 + 37 = 54 + (40 – 3) = (54 + 40) – 3 = 94 – 3 = 91 54 + 37 = (51 + 3) + 37 = 51 + (3 + 37) = 51 + 40 = 91	• Rounding up to the nearest ten, hundred… with adjustment at end to re-establish equilibrium • Equality • Decomposition of number to then re-associate • Associative property • Maintaining the same difference • Order of operations • Zero property: a number minus itself	54 – 37 = 54 – (40 – 3)[8] = (54 – 40) + 3 = 14 + 3 = 17 54 – 37 = (54 + 3) – (37 + 3) = 57 – 40 = 17
Difference Between Also known as: Add Up to Subtract/ Subtract Back To	___	___	• Flexible thinking to restructure a problem • Two constructs of subtraction: take-away & difference between • Relationship between addition & subtraction (Inverse Operation) • Incrementing to next/ previous ten from any number	• 54 – 37 = x transformed into either… 54 + x = 37 or 54 – x = 37 *Typically variations of Incremental Strategies are used to solve the task once restructuring has taken place*

[7] Interestingly, if a child does not know the fact 14 – 7 *yet,* the child's strategy choice is typically limited to counting back by ones. However, if the child has the decomposition skills of using the *Get Back To a Ten* strategy [14 – (4 + 3) = (14 – 4) – 3 = 10 – 3 = 7] he or she can work more efficiently. This decomposition skill is a core idea of the Incremental Strategy. See "Learning the Basic Facts" (Brickwedde, 2012) and "Developing Base Ten Understanding: Working with Tens…" (Brickwedde, 2012) for more details on developing these strategies in children.

[8] As children explore how the Compensation strategies, and the underlying algebraic ideas that govern them, it is more productive to have students think about how the adjustments to the numbers effect the relationships among the quantities in question. In addition, *I added 3 too many so I have to take it off to keep it equal,* is one example. In subtraction, *I took 3 too many away so I need to put 3 back in to make it equal,* also focuses on the effect of the changes that occur. The principles of the order of operations is one means to help make the mathematics visible and can over time become itself a focus of conversation but initially it is important to focus on the effects of change in the relationship among the quantities.