Windows into Children's Mathematical Thinking:

Instructional Tools Focusing on Multiplication & Division in the Intermediate Grades

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Using Children's Thinking To Shape Instruction

The more a teacher knows how a child thinks, the greater the student's achievement. Awareness of how a child thinks shapes the instructional decision-making in which a teacher engages. How does a teacher access that thinking? How is a child's thinking interpreted? What instructional options does that information open up to a teacher? These findings and questions are the basis of the Cognitively Guided Instruction (CGI) professional development program (Carpenter, et. al., 1999). This handout is an attempt to summarize some of the instructional tools identified by the CGI Project, as well as other research projects, that teachers can utilize to gain access to student thinking.

Responding to student thinking is an essential part of the teacher decision-making process in shaping curriculum. A teacher needs to have a clear idea of the "learning trajectory" (Simon, 1995) the students are to be guided through. Comprehending what constitutes an understanding of the mathematics, and what that looks like at various stages of student development, is key knowledge a teacher needs to possess. The analysis of where the students are at in the development of the mathematics is where a teacher begins to select instructional tools, contexts, and number choices that will draw out student thinking and move the students towards a mathematical goal (Hiebert, et al, 1997). Which tools work best and when is not a simple answer. Nor is it always clear which numbers will best draw out student understanding. Teaching itself is a problem-solving activity (Carpenter, 1989). The following summary is an attempt to at least place in context the strengths of various instructional tools that a teacher may draw upon in his or her problem-solving, decision-making process.

Mathematical Big Ideas & Teacher Decision-Making

There are many intermediate-grade learning goals. In the area of number and number operations, however, there is a set of *big ideas* that are essential for which students need to comprehend to gain further fluency and flexibility. Among these big key ideas are:

- Decomposition & reconfiguration of number by various addends, factors, or combinations of factors and addends, e.g, 87 = 8x10 + 7
- Extending and deepening base ten understanding
- Developing multiplicative and proportional reasoning
- Algebraic properties (commutative, associative, distributive, & identity)
- Algebraic concepts of equality, zero, and one

It is around these big ideas that the following instructional tools will be reviewed. It is in this context that a set of teacher decision-making choices is presented. Once the mathematical idea is identified a *set of teacher decision-making questions* can be asked:

- Which instructional tool or combinations of tools best develop the mathematical concept?
- What kind of context would students relate to?
- What kinds of number choices would draw out the mathematics?
- What kinds of questions should be posed to stimulate thinking?
- How might a student be assessed using this tool?
- What constitutes understanding of these big ideas?
- How does a mathematically emergent, developing, or established child engage in this activity?

Instructional Tools

Instructional tools are windows to student thinking. Teachers need to analyze the thinking of their students and continue the instructional decision-making process in light of this new information. Did the students progress in their understanding? The next question, the next task, the next lesson will all be shaped by the teacher's analysis.

Research shows that students engage in the mathematics easier if the mathematics is placed within a context. Context-based instructional tools are a powerful entry point for students to grapple with the nuances of the number system. This does not mean, however, that all mathematics need be contextual in a real-world setting. Mathematics is built on relationships and core concepts. To explore these underlying mathematical ideas, non-context based instructional tools allows the teacher to engage students in essential ideas that transcends context and looks across individual problems to broader algebraic ideas. These non-contextual tools can be equally engaging for students as problem-solving episodes. *It is knowing when to use which tool for which part of the mathematical learning and conversation that marks the decision-making process.*

Context Based Instructional Tools - Problem Types

Context allows students to enter the mathematics more easily. As mentioned previously, this does not mean all mathematics must be in a story context. Further sections will explore non-context based activities. However, *the contextual nature of a problem will often shape the strategies student will utilize. Different problem structures will allow students to* *approach the number system from different angles and perspectives.*

Carpenter, et. al., (1999) outlines eleven different problem structures in addition and subtraction along with three main multiplication and division structures. Where the unknown lies, whether a problem structure is active or non-active, and whether a problem structure is part-whole or comparative in its number relations have explicit developmental implications for young children. Children will spiral through the solution strategies of direct modeling, counting strategies, deriving and flexible strategies, and finally abstract number strategies. Which solution strategy a child utilizes may depend upon the developmental stage of the child, the number size, or complexity of the problem context.

Focusing on multiplication and division explicitly, problem types are divided into three structural categories (*See Figure 1*):

- Asymmetrical problems
  - Equal grouping
  - Price
  - Rate
  - Multiplicative Compare
- Symmetrical problems
  - Area
  - Array
- Combination/Cross Product problems

Many curricula approach multiplication by introducing students to array and area models. The assumption is that students will develop the concept of the commutative property quicker and apply this key concept across all problem contexts. Research does not support this. Students may demonstrate knowledge of the commutative property in symmetrical problems but not transfer that awareness when working with asymmetrical problems. Divorcing the mathematics from the context is very abstract in asymmetrical contexts. It is possible, and an ultimate goal, but not easy.

The instructional implication for teachers is that *students need a variety of contexts in order to develop a deep and robust sense of the range of contexts in which multiplication can be used.* The different contexts allow the student to develop different multiplicative understandings. Rate and price problems are *unit transforming* contexts, e.g., miles per hour or price per pound. Multiplicative comparison problems assist students in developing the iterative aspect of multiplication in which making a composite unit is particularly strong. Seeing multiplication *only* as an array or as an expression of area limits a child's thinking. *Figure 1* notes the variety of multiplication and division problem types.

Non-Context Instructional Tools – True/False Equations; Open Number Equations; Number Strings

Presenting students with an equation such as 42 x 79 = *n* is an example of a non-context math problem. A context can be superimposed upon the numbers - "A storeowner has 42 DVD players to sell. Each costs $79. How much money will the owner have if all the DVD players sell?" Verbally wrapping such number sentences into a context can help students to

Figure 1 - Multiplication and Division Problem Types

Equal Grouping – Asymmetrical Structure

- Multiplication
  There are four basketball teams at the tournament and each team has five players. How many players are at the tournament?
- Measurement or Quotitive Division
  I have 24 apples. How many paper bags will I fill if I put 3 apples into each bag?
- Partitive Division
  Twenty-four apples need to be placed into eight paper bags. How many apples will you put in each bag if you want the same number in each bag?

Rate Problems – Asymmetrical Structure - *(Carpenter, et al. 1999, p. 46)*

- *A baby elephant gains 4 pounds each day. How many pounds will the baby elephant gain in 8 days? (Carpenter, et. al. 1999)*
- *A baby elephant gains 4 pounds each day. How many days will it take the baby elephant to gain 32 pounds?*
- *A baby elephant gained 32 pounds in 8 days. If she gained the same amount of weight each day, how much did she gain in one day?*

Price Problems – Asymmetrical Structure

- *How much would six candies cost if each piece costs 5 cents?*
- *Individual candies costs 5 cents for each piece. How many candies can you buy with 30 cents?*
- *If you can buy 6 pieces of candy with 30 cents, how much does each piece cost?*

Multiplicative Comparison Problems – Asymmetrical Structure

- *James has 16 baseball cards. Deshawn has 5 times as many. How many baseball cards does Deshawn have than James?*
- *Celine read 54 books during summer vacation. Her younger sister Tanya read 9 books during summer vacation. How many times greater is the number of books Celine read compared with the number of books Tanya read?*
- *Jamie read 224 pages of the new Harry Potter book. This is 4 times as many pages as Sydney read? How many pages has Sydney read so far?*

Area and Array Problems – Symmetrical Structure

- *Area - (Carpenter, et. al. 1999, p. 50)*
  - A baker has a pan of fudge that measures 8 inches on one side and 9 inches on the other side. If the fudge is cut into square pieces 1 inch on a side, how many pieces of fudge does the pan hold?
  - A farmer plants a rectangular vegetable garden that measures 6 meters along one side and 8 meters along an adjacent side. How many square meters of garden did the farmer plant?
  - A farmer plans to plant a rectangular vegetable garden. She has enough room to make the garden 6 meters along one side. How long does she have to have to make the adjacent side in order to have 48 square meters of garden?
- *Array - (Carpenter, et. al. 1999, p. 51)*
  - For the second-grade play, the chairs have been put into 4 rows with 4 chairs in each row. How many chairs have been put out for the play?

Combination/Cross Product Problems

Lucy's Pizzeria lists three different meats and four different veggies as toppings. How many different kinds of pizzas consisting of one type of meat and one type of veggie are possible?

access the mathematics. However, presenting numbers in basic equation form can draw out mathematical discussions that are equally as important as when within a context.
Examples of non-context instructional tools are (*see figure 2*):

- True/False Equations
- Open Number Equations
- Number Strings

If a *big idea* is to develop equality, consider the following problem: 4 + 5 = 6 + 4, true or false? Posing this problem and engaging students in an active conversation about whether this is ***true or false,*** and pursing why it is true or false can reveal a child's thinking about the concept of equality. The conversation can also reveal a child's capacity to solve by calculation and/or think relationally across the equal sign (Carpenter, et al, 2003).

Following up with an ***open-number equation*** such as 4 + 5 = *y* + 4 and asking, "What does *y* have to be to make this a true number equation?" extends the conversation to more explicitly consider the issues of equality. Figures 2 and 3 outlines various ways each of these tools can be utilized to draw out different mathematical ideas and, therefore, extend students' comprehension of those ideas.

Number Strings, also referred in some materials as *Mental Math Strings* (Fosnot & Dolk, 2001), *Problem Stems* (Carpenter, et al, 1999), or *series of equations* (Carpenter, et al., 2003) are a series of mathematical expressions or equations presented to students one at a time. The notion is to create a series of expressions connected around a mathematical idea the students can explore. If a goal is to have students develop *deriving strategies* – strategies that use properties of operations or algebraic thinking – the following sequence might be useful: 5 x 7; 6 x 7; 4 x 7; 8 x 7. The conversation around such a string would sound like:

"What are five groups of seven?"

"If you know what five sevens are, can you use that to figure out what are six groups of seven?"

If a child indicates that they used 5 x 7 to figure out 6 x 7, the mathematical representation the teacher can draw out on the board or chart paper could be 6 x 7 = (5 x 7) + (1 x 7). Conversationally, this may sound like,

"So you are telling me that six sevens are the same as five groups of seven plus one more group of seven? Is that true or false?"

By writing out the mathematical representation in this manner, the student is able to consider several mathematical ideas. Among these are that six can be decomposed into 5 + 1, the distributive property can be used to solve the problem, and that both sides of the equation are of equal value. The string of expressions continues to grow as you ask the children to use what happened in previous combinations to solve the new one.

One must be alert, however, that there will be a student who is thinking more divergently. Using 6 x 7 again, it is reasonable to expect that a child would not refer back to 5 x 7 as you planned but rather use another legitimate strategy. Such a scenario might go as follows:

S. "I did three sevens are 21, then I just doubled it."

Figure 2 - True/False Statements

True/False equation can be organized in several different ways:

- Basic true and false sentences

  3 + 5 = 8   7 + 4 = 10 F

- Equations requiring computation (forces a comparison of calculated results across the equal sign)

  57 – 38 = 19   94 – 38 = 46 F

- Equations that encourage estimation strategies (Generally false statements)

  38 + 79 = 207 F   24 x 38 = 252 F

- Number equations that encourage students to focus on particular aspects of a calculation (Can be used to help sort through number sense ideas or areas of computational difficulty)

  280 + 456 = 734 F (The unit's digits are unequal)
  24 x 357 = 8569 F (The product of an even number and an odd number is an even number)
  237 + 40 = 277
  40 x 100 = 400 F

- Sequences of number equations (As an example, the following sequence provides an opportunity to consider the effect of adding or subtracting a number of one of the numbers in a subtraction problem.)

  347 – 258 = 89
  327 – 258 = 69
  337 – 258 = 59 F

  (Sequences might also encourage children to look at relations between number facts or the relation between addition and subtraction)

  6 + 6 = 12   258 + 89 = 347
  6 + 7 = 13   347 – 89 = 258

- Basic Properties (As a means of initiating discussions around basic properties of number)

  3,798 + 0 = 3,798   27 – 0 = 27
  4,567 – 32 = 4,567 F   0 – 27 = 27 F
  2 + 8 = 8 + 2   56 x 1 = 57 F

- Equality as a relation (To provide a context for discussing equality as a relation as well as looking explicitly at the properties of equality)

  3 + 8 = 2 + 9   19 – 15 = 14 - 10
  356 + 247 + 98 = 356 + 247 + 98   9 = 9
  (Reflexive Property)
  56 + 27 = 100 – 17   100 – 17 = 56 + 27
  25 x 6 = 150   150 = 25 x 6
  (Symmetric Property)

T. "Where did the three and the two come from?"
S. "I broke the 6 into 2 x 3."
T. "So, you are telling me (writing on the board) that 6 x 7 = (2 x 3) x 7?"
S. "Yes"
T. "And that you multiplied [continuing the equation chain begun above] = 2 x (3 x 7) = 42?"
S. "Yes."
T. [To the class] "Is this strategy the same or different from the first strategy? [Where 6 was broken into 5 + 1]

Here the student decomposes the number into factors (6 = 2 x 3) and then uses the associative property to create an "easier" combination, (2x3) x 7 = 2 x (3 x 7). In such a scenario, the teacher needs to respond to the strategies students bring forward and use those strategies to develop various mathematical relations.

These three non-context instructional tools can be mixed and matched to help shape conversations. A true/false equation might be interjected into a number string in order to help a student clarify a statement. An open-number equation might help students extend a discussion to a different set of numbers in order to reinforce an idea that was raised through discussion. Inherent with all three of these techniques is the reinforcement and development of the algebraic concept of equality, decomposition of number, and core algebraic properties such as the distributive and associative properties. Figures 2, 3 & 4 summarize the variety of conditions for which true/false and open-number equations, and number strings might be effective.

Summary

Achievement is linked to how well a teacher understands how a student thinks and the mathematics embedded in the strategies. To establish an environment where a student can flourish, the teacher must first gain access to his or her students' thought processes. A teacher must, therefore, work to create windows into children's thinking. The tools selected, the numbers chosen, the prompts and questions asked, and the verbalization of ones strategies all merge to create an environment in which a child can express his or her thinking. The teacher is not a passive player in this instructional setting. Rather, the teacher is the key organizer and shaper of the events in the classroom. It is knowing which tool to use and when to use it that helps shape student thinking around specific mathematical ideas. The environment created within such classroom communities leads to both a robust understanding of the mathematics while developing the habits of a skilled mathematician.

References

In addition to the references listed below, the information listed in the figures 2 & 3 draw directly from Integrating Arithmetic and Algebra in the Elementary School course handouts (2000) created by Carpenter and Levi and used here by permission

To learn about children's solution strategies and problem types:

- Carpenter, T. P., E. Fennema, M.L. Franke, L. Levi, S.B. Empson (1999). *Children's Mathematics: Cognitively Guided Instruction.* Portsmouth, NH: Heinemann.

To learn about generalizing arithmetic to algebraic concepts and principles and non-context instructional techniques:

- Carpenter, T. P., M.L. Franke, L. Levi (2003) *Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School.* Portsmouth, NH: Heinemann Press.

To learn about features of problem-solving based classrooms from four research projects:

- Hiebert, J., T.P. Carpenter, E. Fennema, K.C. Fuson, D. Wearne, H. Murray, A. Olivier, P. Human. (1997). *Making Sense: Teaching and Learning Mathematics with Understanding.* Portsmouth, NH: Heinemann.

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- Carpenter, T. P. (1989). Teaching as problem solving. In R. I. Charles & E. A. Silver (Eds.). *The teaching and assessing of mathematical problem solving* (pp. 187-202). Reston, Va.: Erlbaum Associates.
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- Fosnot, C.T., M. Dolk. (2001). *Young Mathematicians at Work: Constructing Multiplication and Division.* Portsmouth, NH: Heinemann.
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Figure 3 - Open-Number Equations

Open-number equations, too, can be designed to explore or draw out specific mathematical ideas depending upon how number choices and operations are juxtaposed.

- Basic open number equations (Used to represent basic addition, subtraction, multiplication, or division word problems, and children may solve them using the same strategies that they would use to solve the corresponding word problems.)

  $3 + c = 8$ $12 - t = 4$ $r \times 4 = 32$

- Equations involving more than a single operation

  $c + c + 5 = 13$ $2 \times c + 5 = 13$

- Equations that support the learning of arithmetic skills

  √ Relations among number facts
  $6 + 8 = c + 10$
  $6 \times 8 = (5 \times 8) + (t \times 8)$
  $8 \times 6 = 8 \times 5 + m$
  √ Base-ten number concepts
  $84 = v + 4$
  $94 = 84 + s$
  $56 + p = 86$

- Equations that support the generation of conjectures
  $578 \times g = 578$
  $497 - m = 0$
  $598 + 476 = 476 + t$

- Open number equations with multiple variables
  $r + s = 7$

- Open number equations with repeated variables
  $w + w = 20$
  $v + v + v - v = 24$
  $v + v + v + v = v + v + 18$

Figure 4 - Mental Math Strings

- Developing derived strategies for learning the basic facts
  8 x 8
  8 x 9
  8 x 7
- Developing of specific strategies
  The effect of doubling and halving
  3 x 24 124 ÷ 12
  6 x 12 62 ÷ 6
  Using the distributive property
  6 x 6
  6 x 8
  6 x 14