Windows into Children's Mathematical Thinking:

Problem Types, Levels of Development & Children's Solution Strategies: Addition, Subtraction, Multiplication, & Division

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Observing Children's Thinking

Angelina was playing with her stuffed animals on the floor of her bedroom. When it was time to clean up, she put three of the stuffed animals on the bed. How many more does she need to put up on her bed to have all seven of her stuffed animals off the floor?

Angelina, a beginning first grader, takes three cubes and places them on the table in front of her. She then starts counting as she puts more cubes in front of her saying, "four, five, six, seven." In the counting, she keeps the new cubes just a little apart from the others. She answers four more stuffed animals need to be picked up.

Ari does something similar but announces his answer to be seven. You had observed him count out three cubes then count on to seven but there is only a single pile of cubes in front of him. What, besides the separate answers, is different between these two children? How are you as the teacher to use this information?

Consider the following exchange between the two students and the teacher.

T: Ari, I noticed you used the cubes. Tell me how you solved the problem.
Ari: I counted three stuffed animals and then I counted four, five, six, seven. The answer is seven.
T: Angelina, tell me what you did with your cubes.
Angelina: I put out three cubes then went four, five, six, seven. I am pretending these are the stuffed animals I put up second so my answer is four more stuffed animals?
T: Angelina, so what are you pretending this first group of cubes are?
Angelina: These are the stuffed animals that I put up first.
T: And what are you pretending these cubes are? (Pointing to the second group.)
Angelina: These are the stuffed animals that I put up second.
T: I noticed you kept those cubes separate from the first group. How did that help you solve the problem?
Angelina: Because if I didn't keep them separate I wouldn't know what my answer was.
T: Ari, what did Angelina do with her cubes that is different from the way you used yours?

Depending upon Ari's response, you can determine whether Ari's issue is one of developmental understanding or one of organizational confusion. You also observed watching your students that Jerry worked with his fingers. All you heard him say to himself was, "four, five, six, seven; (extending one finger with each count) the answer is four." A public conversation with Jerry reveals that he just pictured the first group of stuffed animals in his head. He didn't need to count them out since they were already on the bed.

Depending upon your own upbringing, and depending upon your familiarity with how children solve an instructional task like this, you might be scratching your head wondering why no child took out seven cubes and removed three of them to find the answer. That was how you were taught as a child to do this type of scenario.

The following sections focus attention on these big ideas:

1. The differences in structure among problem types
2. Developmental responses to the different problem types
3. Developmental progressions in children's solution strategies in solving the problem types

Children's Response to the Structure of Problem Situations

Researchers began in the early 1980s to look closely at how young children approached various problem *structures* (Carpenter & Moser, 1984). In that work, several important instructional implications were identified:

1. There is a developmental progression in terms of ease of solving problems among the different problem types.
2. There is a developmental progression in terms of increasing abstraction in the types of solution strategies that children use to solve the various problem types.
3. Action in the context of the problem makes some problem types easier than others.

As the research expanded, these patterns held across all four operations of addition, subtraction, multiplication, and division. Another important aspect of this early research was that children as young as kindergarten can solve a wide range of these problem types because they can be modeled in ways that are natural to them and tap into their informal knowledge (Carpenter, et al., 1993). For teachers, the question arises of how can we draw out this informal knowledge of young children and use that knowledge as a conduit for children to develop important mathematical ideas.

Problem Types

To begin to sort through the differences among the problem types, think about two things: the *action or non-action* of the *context,* and what is *unknown* (for addition and subtraction) in the situation or what is *known* (for multiplication and division). The labels used to describe the addition and subtraction problem types come from the action/non-action implied in the context. First let us examine the difference between active and non-active scenarios.

Action vs. Non-Action in Context

Imagine you are videotaping a scene. *Robbie has picked 6 strawberries and placed them into his basket. Then he picks 7 more strawberries.* In the storyboard for this video clip, the viewer sees 6 strawberries already in the basket. Then the main character, Robbie, adds more to the basket. The basket fills with more strawberries. *There is change over time.*

Now imagine videotaping another scene. *Robbie has two baskets of fruit. There are 6 strawberries in one basket and 7 apples in another.* At the beginning of the video, the baskets have 6 strawberries and 7 apples. At the end of the movie, the baskets still have 6 strawberries and 7 apples. *Nothing changed* over the course of the video.

Mathematically, I can ask questions to further each of these scenes. In the former I could ask, "How many strawberries did Robbie end up picking?" For the latter I could ask, "How many pieces of fruit does Robbie have altogether?" The mathematical structure for both of these situations would look the same: $6 + 7 = y$. What the researchers found, however, is that the first scenario is easier for young children to solve than the second. The act of ***joining to***[1]*,* or its counterpart ***separating from****,* makes more initial developmental sense than ***part-part-whole****,* discrete scenarios. The action of picking, giving, taking, losing, getting are easier to visualize, access, and role play for young children and even for English language learners. The "doing," the change over time implied in the context, makes these scenarios more accessible to children.

The Unknown

Mathematically I can change where the unknown is located in a problem. It changes the question that is asked of the problem solver. Consider the equation $6 + 7 = 13$. In the two scenes described above, the mathematical structure was $6 + 7 = y$. The *result is unknown* (active) or the *total is unknown* (non-active). But what if the information sought is different? *Robbie has 6 strawberries. How many must he pick to have 13?* The mathematical structure for this problem is $6 + y = 13$. The *change* is *unknown* (active).

The third place where the unknown can occur is when the *start* is *unknown* (active). In this scenario, *Robbie had some strawberries in his basket. He picked 7 more. Now he has 13. How many did he have to start with?* The mathematical structure for this is $y + 7 = 13$.

Recap

Active contexts are more accessible for early learners than non-active. Children's modeled solution strategies look different as well. To categorize the problem situations into active and non-active, the following holds:

Active	Non-Active
Joining	Part-Part-Whole
Separating	Comparing

Unknown Descriptors

Result unknown	Whole unknown
Change unknown	Part unknown
Start unknown	Difference unknown

Developmental Implications

Where the unknown is located influences the developmental difficulty of the task for a child. The easiest problems to solve developmentally are where the result or the whole is unknown, whether or not the context is a joining, separating, or part-part-whole. This is because no planning is required of the child. They work in order with the first step, do the next step, and find the total.

Returning to Angelina and Ari and their work described at the beginning, to be successful with the change unknown problem, Angelina had to *plan ahead* and set apart the new cubes from the original set of cubes she had put out in order to know which group would be her answer. Ari, who had an idea of what was being asked in the task, did not plan ahead and so was not able to "see" his answer. All he saw was the combined set and so answered "7."

Three developmental clusters form among the various problem types. The first in the list is the "gate keeper." If a child can't do the first problem it is less likely that they can do the others. Generally speaking, addition is easier than subtraction, active is easier than non-active problem types.

Easiest
- Join, Result Unknown
- Separate, Result Unknown
- Part-Part-Whole, Whole Unknown
- Multiplication*

Middle Level
- Join, change unknown
- Separate, change unknown
- Multiplication*
- Compare, Difference Unknown
- Measurement Division
- Partitive Division

Hardest Level
- Join, Start Unknown
- Separate, Start Unknown
- Part-Part-Whole, Part Unknown

* Some evidence suggests that for *some* children multiplication comes earlier than the change unknown problems. Therefore it is considered a cusp problem.

Children's Solution Strategies

The original research around children's response to the problem types identified a progression in the solution strategies children use. The two clusters noted above that are the easiest and middle developmental levels are accessible to children because they can all be *directly modeled.* This key term is essential to understand.

Direct Modelers are *locked* into following the structure of the problem. Modeling includes concrete representations of all sets involved (manipulative or pictorial). Typical of early direct modelers is, if interrupted, they need to go back to the beginning and restart the process. An example of a direct modeler was Angelina in the opening vignette. She represented in cubes a starting set of 3, then using blocks counted on to seven. If she does not instantly recognizes the new set she would have to recount that second set to determine that there were 4, thus giving her the answer.

Jerry is a ***counter.*** His strategy was to hold the first set in his head and then count on using fingers to keep track of the set. This represents the next level of abstraction children demonstrate. Still following the structure of the context, he counted on to seven to find his total. The first set of three was mentally abstracted.

If you saw a child, even if they used cubes, take out seven and remove three, then count the remainder, that child is a ***flexible thinker***. This child has *temporally restructured* the problem from being $3 + y = 7$ to $7 - 3 = y$. Using the video storyboard analogy once again, this child runs the tape in reverse. This is very abstract for children to comprehend and reflects the emergence of key algebraic ideas.

The most abstract developmental level is ***derived & number facts.*** The hallmark of derived facts is the emergence of ***relational thinking***, another key aspect of algebraic thinking. With 3 + 4, the child may know that 3 + 3 = 6 so 3 + 4 = 7. The child uses what is known about a familiar combination to relationally calculate the unknown. The properties of operations, specifically the associative property, is utilized Intuitively. [3 + 4 = 3 + (3 + 1) = (3 + 3) + 1 = 6 + 1] Finally, a child knows the answer is 4 because 3 + 4 = 7 is a known combination. At the number level, the child's computational thinking is done mentally with only those items visible that the child needs to free up working memory to complete the execution of the equation.

While there is a progression of *direct modeling, counting, flexible thinking,* and *derived & number fact levels,* it is not necessarily a linear one. Number size can effect which level solution strategy one uses. A child can be a fluid counter with small numbers but may need to direct model multidigit numbers. A child may have automaticity with 6 + 6 but has to count on for 6 + 8. The progressions are more fluid and overlapping. Over time the goal is to be more abstract in ones thinking applying the properties of operations and thinking relationally. An important message for students to learn, however, is that the modeling processes are powerful mathematical tools that should be readily used and not to be looked down upon.

Child's Logic vs. Adult Logic – Being a Flexible Thinker

All of us learned certain solution strategies to particular problem structures growing up. We have already looked at Angelina's solution to the join, change unknown structure. We were taught to subtract with this scenario. But children don't see "you have three, how much to get to 7?" as a subtraction structure. This is an addition context for children. Yes, subtraction is a viable strategy to Angelina's task, but it is not necessary, nor necessarily more advantageous to do so.

Another type that needs a close look is compare, difference unknown. *"You have 7 balloons, I have 12. How many more do I have than you?"* As a child, there is a high likelihood that you were told to subtract to solve this problem. But look at how children perceive this problem.

A direct modeler uses a *matching* strategy to solve this. The child lines up a row of 7 cubes and a row of 12 cubes, then matches the cubes one-to-one. The answer is the unmatched cubes. When you observe what the child is doing, he or she is neither adding nor subtracting. To solve this problem any other way, a child *must be* a flexible thinker.

There are *three choices* that can be made here, all mathematically legitimate. One can restructure the problem into a join, change unknown and count up ($7 + y = 12$). One can restructure the problem into a separate, change unknown and count back to 7 ($12 - y = 7$). Finally, one can do what we were told to do growing up and "take away" 7 from 12 ($12 - 7 = y$). Number choices may drive the decision to choose one strategy over another. With the numbers 7 and 12, it is less work if you are a counter, meaning more efficient, to either add up to 12 or count back to 7. The child doing either strategy would use only five fingers. Two more fingers have to be used with these numbers to take away seven.

The efficiency of strategy changes, however, if the numbers were 5 and 12. Now 12 – 5 uses fewer fingers than $12 - y = 5$. Being flexible and efficient in ones thinking is not just being able to restructure a problem. It is being able to choose among legitimate strategies to select the best option.

The Kindergarten Study – Children's Informal Knowledge

In a published study (Carpenter, et al., 1993), researchers reported high percentages of kindergarteners could solve all the problems discussed here. These were young children who had been presented problems all year long. They were not "taught," meaning explicitly shown, how to solve a problem in any particular manner. The kindergarteners used strategies that made sense to them and openly discussed those strategies with classmates. Some kindergartens still had some one-to-one counting errors but the strategies on how to solve the problems were clearly present. By the spring of the year, 87.1% could use a correct strategy to solve a multiplication problem; 80% the change unknown type problem. 72.9% of the students solved the compare problem, with 74.3% solving a measurement division problem. The reason for this success is that all of these problems are accessible to kindergarteners through the direct modeling process and they solved the problems using logic that made sense to them.

Summary of Problem Types With Solution Strategies

• **Join, Result Unknown** (*Add To, Result Unknown*): ($5 + 7 = y$) *Ann has five pennies. She gets 7 more pennies from her dad. How many pennies does she have now?* **Direct Modeler:** Makes a pile of 5 cubes and 7 cubes, joins them all and counts all starting from one. **Counter:** Says five, then keeping track on her fingers counts, 6, 7, 8, 9, 10, 11, 12. Counts fingers to find answer. Counts from the larger: Counts on from 7 to save time. **Relational Thinker** (Derived Strategy): 5 + 5 = 10, 10 + 2 = 12.

• **Separate, Result Unknown** (*Take From, Result Unknown*): ($12 - 7 = y$) *Ann has 12 pennies in her hand. She puts 7 pennies in her pocket. How many pennies does she still have in her hand***?** **Direct Modeler:**

Pile of 12 cubes, separates 7, counts remaining cubes. **Counter**: Counts back from 12 until seven fingers are up. Answer is 5. **Flexible Thinker:** Restructures into a join, change unknown problem. Counts up from 7 to 12. **Relational Thinker:** 12 – 2 = 10, 10 – 5 = 5.

• **Join, Change Unknown** (*Add To, Change Unknown*)**:** (5 + y = 12) *Abdu has 5 pennies. How many more does he need to have 12 pennies?* **Direct Modeler:** A pile of 5 cubes, counts on to 12 keeping new set of cubes separate from the original. Counts the new cubes to find answer. **Counter**: Counts on from 5 extending a finger with each count. Number of fingers is the answer. **Flexible Thinker:** 12 – 5 = 7. **Relational Thinker:** 5 + 5 = 10, 10 + 2 = 12, 5 + 2 = 7.

• **Separate, Change Unknown** (*Take From, Change Unknown*): (12 – y = 7) *Abdu has 12 pennies. He gives some of the pennies to his cousin. Now he has 7 pennies. How many pennies did he give his cousin?* **Direct Modeler:** Makes pile of 12 cubes. Removes cubes until five are left. Counts cubes that were removed. **Counter**: Starts at 12, counts back stopping at 7. Counts number of fingers used. **Flexible Thinker:** Restructure into a join, change unknown and count up, or into a separate, result unknown and subtract 7 from 12. **Relational Thinker:** 12 – 2 = 10, 10 – 3 = 7, 2 + 3 = 5.

• **Compare, Difference Unknown** - (no number sentence specifically reflects this problem structure) *Lilly has 7 pennies. Emil has 12 pennies. How many more does Emil have than Lilly?* **Direct Modeler:** A train of 7 cubes matched one-to-one with a train of 12 cubes. Counts the unmatched cubes. **Counter**: There is no counting strategy to match this problem type. **Flexible Thinker:** Restructure the task into 7 + y = 12, 12 – y = 7, or 12 – 7 = y. Uses counting, relational, or fact recall to solve. **Relational Thinker:** 7 + 3 → 10 + 2 → 12, or 12 – 2 → 10 – 3 → 7, or 12 – 2 → 10 – 5 → 5.

• **Join, Start Unknown** (*Add To, Start Unknown*)**:** (y + 5 = 12) *Arthur has pennies in his pocket. He puts 5 more in his pocket. Now he has 12. Now many pennies were in his pocket to start with?* **Direct Modeler:** Trial & error. **Counter:** There are no counting strategies for this problem situation. **Flexible Thinker**: Restructure the problem into 5 + y = 12, 12 – 5 = y, or 12 – y = 5. Use modeling (not direct modeling)[2], counting, or relational thinking to solve. **Relational Thinker:** See previous examples that match the structure.

• **Separate, Start Unknown** (*Take From, Start Unknown*)**:** (y – 5 = 7) *Arthur had some pennies. He lost 5 of them. Now he has 7 left. How many did he have to start with***?** **Direct Modeler:** Trial & error. **Counter**: There are no counting strategies for this problem type. **Flexible Thinker**: Restructure the problem into 5 + *7 = y or* 7 – 5 = y. Uses modeling (not direct modeling), counting, relational strategies to solve the problem. **Relational Thinker:** See previous examples that match the structure.

• **Multiplication:** (3 x 5 = y) *Juquila has three piles of pennies with 5 pennies in each pile. How many pennies does she have altogether?* **Direct Modeler:** Puts cubes into 3 piles with five in each pile. Counts all of the cubes starting from one. **Counter**: Skip counts by fives to fifteen. **Relational Thinker**: 2 x 5 = 10 so add five more.

• **Measurement Division:** (15 ÷ 3 = p) *Luke has 15 pennies. He wants to give three pennies to his friends. To how many friends can Luke give three pennies?* **Direct Modeler:** Create a pile of 15 cubes. Start passing out 3 into groups until there are no more cubes left. Count the piles. **Counter**: Skip count by three to fifteen. **Flexible & Relational Thinkers:** p x 3 = 15.

• **Partitive Division** – (15 ÷ 5 = p) *Liz has 15 pennies that she wants to give to 5 friends. If she gives the same number of pennies to each friend, how many pennies will each friend get?* **Direct Modeler:** Makes a pile of 15. Starts passing out one cube to each of five piles. Continues to pass out the cubes one by one until all cubes are gone. Count the number of cubes in each pile. **Counter**: Trial & Error. **Flexible & Relational Thinkers:** 5 x p = 15.

Compare, Quantity Unknown *(Compare, Bigger/Smaller) Tom has 17 pennies. Eliz has 3 less than Tom. How many pennies does Eliz have?* Or, *Tom has 17 pennies. Eliz has 3 more than Tom, How many pennies does Eliz have?* **Direct Modeler:** There is no direct modeling construct for this problem type. **Counter:** There is no counting construct for this problem type. **Flexible Thinker:** One must be a flexible thinker to solver this problem type. For the first scenario, the problem could be restructured into 17 – 3, for the second scenario 17 + 3. If the combinations were *Tom had 17, that's 12 more than Eliz,* the problem would more likely be transformed into 17 – y = 12 or 12 + y = 17.

Compare, Referent Unknown *(Compare, Bigger/Smaller) Tom has 17 pennies, that's 3 more than Eliz. How many pennies does Eliz have?* Or*, Tom has 17 pennies. That's 3 fewer than Eliz. How many does Eliz have?* **Direct Modeler:** There is no direct modeling construct for this problem type. **Counter:** There is no counting construct for this problem type. **Flexible Thinker:** One must be a flexible thinker to solver this problem type. For the first scenario, Tom has 3 more than Eliz, he problem can be transformed into 17 – 3 = y (given these numbers most likely) or 17 – y = 3. For Tom has 3 fewer than Eliz, 17 + 3 = y is likely. Number combinations could also elicit other restructured strategies as noted with the quantity unknown task.

Nuances and variations might occur with each of the problems listed. Generally, however, most fit within the patterns noted. If you observe something different, make notes and talk over what was witnessed with a colleague. Ask questions. Is the structure of the problem being followed? What is the child pretending a particular cube represents?

Try some of these problems with your students. Put you own student names in, adjust the numbers, observe and ask questions.

References

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[1] The labels for the problem types and strategy descriptions in this article reflect the language developed in the research now as Cognitively Guided Instruction. New labels are emerging with the adoption of the Common Core Standards for Mathematics. These terms appear later in parentheses in the summary of problem types. A companion piece connects the CGI labels with those of CCSM.

[2] It is not the use of tools or pictures that determines a direct modeler. It is whether the child is *locked into following the structure* of the problem (direct modeler) or has the abstract capacity to *reordered the sequence of events* (flexible thinker). So a child in this scenario is using modeling to solve the restructured sequence he or she has selected.